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CHAMBERS'S EDUCATIONAL COURSE,—EDITED BY W. AND R. CHAMBERS.

SOLID AND SPHERICAL GEOMETRY,

AND

CONIC SECTIONS.

BEING A TREATISE ON THE HIGHER BRANCHES OF SYNTHETICAL GEOMETRY, CONTAINING THE SOLID AND SPHERICAL GEOMETRY OF PLAYFAIR;

THE PROJECTIONS OF THE SPHERE AND CONIC SECTIONS OF WEST;

WITH PERPENDICULAR PROJECTION AND PERSPECTIVE, AND VARIOUS IMPROVEMENTS AND ADDITIONS.

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EDINBURGH:

PUBLISHED BY WILLIAM AND ROBERT CHAMBERS;

AND SOLD BY ALL BOOKSELLERS.

1837.

EDINBURGH: PRINTED BY W. AND R. CHAMBERS,
19, WATERLOO PLACE.

PREFACE.

This volume contains the Higher Branches of Synthetical Geometry. It consists of Treatises on Solid Geometry, Spherical Geometry, Spherical Trigonometry, the Projections of the Sphere, Perpendicular Projection, Linear Perspective, and Conic Sections.

The first three Treatises are those contained in Playfair's Edition of Euclid's Elements, with some alterations. Several useful Definitions, Scholia, Corollaries, and Propositions, have been added. Instead of the first four propositions on Spherical Geometry, other four, from the excellent System of Mathematics by West, have been substituted, as they contain a more full exposition of principles. From the same work another important proposition, the eighteenth, has been added. The Spherical Trigonometry has been improved by inserting the twelfth and thirteenth, for other propositions which are somewhat intricate. These two are also taken from West's system, but the expressions in the demonstrations are considerably altered. The propositions which these have displaced, rather injure the symmetry of the system, as they require the aid of Analytical Trigonometry. A second demonstration has been given of the fifth proposition of the second book of Solid Geometry, depending on the principle established in the twenty-seventh proposition of the additional fifth book of the former volume.

The two Treatises on the Projections of the Sphere are also adopted from West's system, except the problems on the Stereographic Projection, which, with the Treatises on Perpendicular Projection and Perspective, have been composed expressly for this work.

The Treatise on Conic Sections is taken from the same work; and as the Corollaries are numerous, and most of them important, but undemonstrated in the above work, except a few, demonstrations have been added to those that required them, in order to remove unnecessary obstacles to the progress of the student, who will find a sufficient field for exercise in the undemonstrated theorems and problems annexed for this purpose to this Treatise.

The Treatises on Projections have been added on account of their utility in some branches of practical science and of art; the Projections of the Sphere being necessary in Spherical Trigonometry, and in Nautical and Practical Astronomy; and Perpendicular Projection and Perspective being indispensable in constructing the diagrams in Geometry of three *dimensions*, and *figures* of objects in various branches of the arts and of *philosophy*.

PREFACE.

In the Treatise on Solid Geometry are found several examples of the use of the Method of Exhaustions employed by the ancient geometricians. For some remarks on this subject, and on a particular rule observed by Euclid in the composition of his geometry, the following quotation from the preface to Professor Playfair's treatise is subjoined:—

"With respect to the Geometry of Solids, I have departed from Euclid altogether, with a view of rendering it both shorter and more comprehensive. This, however, is not attempted by introducing a mode of reasoning looser or less rigorous than that of the Greek geometer; for this would be to pay too dear even for the time that might thereby be saved; but it is done chiefly by laying aside a certain rule, which, though it be not essential to the accuracy of demonstration, Euclid has thought it proper, as much as possible, to observe.

The rule referred to is one which regulates the arrangement of Euclid's propositions through the whole of the Elements, namely, that in the demonstration of a theorem he never supposes any thing to be done, as any line to be drawn, or any figure to be constructed, the manner of doing which he has not previously explained.

In the two Books on the Properties of Solids that I now offer to the public, though I have followed Euclid very closely in the simpler parts, I have nowhere sought to subject the demonstrations to such a law as the foregoing, and have never hesitated to admit the existence of such solids, or such lines as are evidently possible, though the manner of actually describing them may not have been explained. In this way, also, I have been enabled to offer that very refined artifice in geometrical reasoning, to which we give the name of the Method of Exhaustions, under a much simpler form than it appears in the twelfth book of Euclid; and the spirit of it may, I think, be best learned when it is disengaged from every thing not essential to it. That this method may be the better understood, and because the demonstrations that require it are, no doubt, the most difficult in the Elements, they are all conducted as nearly as possible in the same way through the different Solids, from the pyramid to the sphere. The comparison of this last Solid with the cylinder concludes the eighth book, and is a proposition that may not improperly be considered as terminating the elementary part of Geometry."

This volume, with the preceding (the ELEMENTS of PLANE GEOMETRY), forms a sufficiently extended Elementary Course of Synthetical Geometry. The higher principles of Trigonometry, and the more abstruse properties of Curves, are fully and clearly investigated by the only adequate method, which is founded on Algebraical Analysis, the application of which to these subjects constitutes the branches of Analytical Trigonometry and Analytical Geometry.

EDINBURGH, September 1, 1837.

ELEMENTS OF SOLID GEOMETRY.

FIRST BOOK.

Solid Geometry treats of the properties of geometrical figures existing in space. Hence, these figures possess extension in the three dimensions of length, breadth, and thickness; they do not therefore exist in the same plane, but they may be represented by means of diagrams drawn on a plane.

DEFINITIONS.

1. A straight line is *perpendicular*, or at *right angles* to a plane, when it makes right angles with every straight line meeting it in that plane.

2. A plane is *perpendicular* to a plane, when the straight lines drawn in one of the planes perpendicularly to the common section of the two planes, are perpendicular to the other plane.

3. The *inclination* of a straight line to a plane is the acute angle contained by that straight line, and another drawn from the point in which the first line meets the plane, to the point in which a perpendicular to the plane drawn from any point of the first line, meets the same plane.

4. The *inclination* of a plane to a plane is the acute angle contained by two straight lines drawn from any the same point of their common section at right angles to it, one upon one plane, and the other upon the other plane.

5. Two planes are said to have the *same*, or a *like inclination* to one another, which two other planes have, when their angles of inclination are equal to one another.

6. Parallel planes are such as do not meet one another though produced.

7. A straight line and plane are *parallel*, if they do not meet when produced.

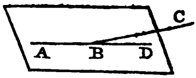
8. The angle formed by two intersecting planes is called a *dihedral* angle.

9. Any two angles are said to be of the same *affection*, when they are either both greater or both not greater than a right angle. The same term is applied to arcs of the same or equal circles, when they are either both greater or both not greater than a quadrant.

PROPOSITION I. THEOREM.

One part of a straight line cannot be in a plane, and another part above it.

If it be possible, let AB, part of the straight line ABC, be in the plane, and the part BC above it; and since the straight line AB is in the plane, it can be produced in that plane; let it be produced to D. Then ABC and ABD are two straight lines, and they have the common segment AB; which is impossible (Pl. Ge. I. Def. 3, Cor.)* Therefore ABC is not a straight line.

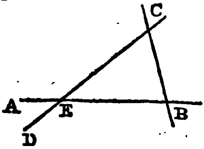


PROPOSITION II. THEOREM.

Any three straight lines which meet one another, not in the same point, are in one plane.

Let the three straight lines AB, CD, CB, meet one another in the points B, C, and E; AB, CD, CB, are in one plane.

Let any plane pass through the straight line EB, and let the plane be turned about EB, produced, if necessary, until it pass through the point C. Then, because the points E, C, are in this plane, the straight line EC is in it (Pl. Ge. I. Def. 8); for the same reason, the straight line BC is in the same; and, by the hypothesis, EB is in it;



* *Pl. Ge.* refers to the volume on Plane Geometry.

therefore the three straight lines EC, CB, BE, are in one plane; but the whole of the lines DC, AB, and BC, produced, are in the same plane with the parts of them EC, EB, BC (I. 1.) Therefore AB, CD, CB, are all in one plane.

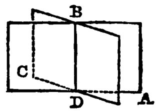
COROLLARY 1.—It is manifest that any two straight lines which cut one another are in one plane.

COR. 2.—Only one plane can pass through three points, or through a straight line and a point; and these conditions therefore are sufficient to determine a plane.

PROPOSITION III. THEOREM.

If two planes cut one another, their common section is a straight line.

Let two planes AB, BC, cut one another, and let B and D be two points in the line of their common section. From B to D draw the straight line BD; and because the points B and D are in the plane AB, the straight line BD is in that plane (Pl. Ge. I. Def. 8); for the same reason, it is in the plane CB; the straight line BD is therefore common to the planes AB and BC, or it is the common section of these planes.

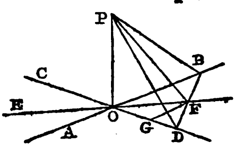


PROPOSITION IV. THEOREM.

If a straight line stand at right angles to each of two straight lines in the point of their intersection, it will also be at right angles to the plane in which these lines are.

Let PO be perpendicular to the lines AB, CD, at their point of intersection O, it is perpendicular to their plane.

For, draw through O any straight line EF in their plane. In OD take any point G, and make $GD = OG$, and through G draw GF parallel to OB, to meet OF in F; join DF, and produce DF to meet AB in B, and join PD, PF, and PB.



Because $OG = GD$, therefore (Pl. Ge. VI. 2.) $BF = FD$; and because, in triangle DOB, the side DB is bisected in F, therefore (Pl. Ge. II. A.) $DO^2 + OB^2 = 2OF^2 + 2FB^2$.

And for a similar reason, in triangle DPB, $DP^2 + PB^2 = 2\,PF^2 + 2\,FB^2$. But since POD is given a right angle, therefore (Pl. Ge. I. 47) $PD^2 = PO^2 + OD^2$; and for a similar reason $PB^2 = PO^2 + OB^2$. Therefore $PD^2 + PB^2 = 2\,PO^2 + OD^2 + OB^2 = 2\,PO^2 + 2\,OF^2 + 2\,FB^2$, for it was shown that $OD^2 + OB^2 = 2\,OF^2 + 2\,FB^2$. But it was also proved that $PD^2 + PB^2 = 2\,PF^2 + 2\,FB^2$; and therefore $2\,PF^2 + 2\,FB^2 = 2\,PO^2 + 2\,OF^2 + 2\,FB^2$; or, taking $2\,FB^2$ from both, $2\,PF^2 = 2\,PO^2 + 2\,OF^2$, or $PF^2 = PO^2 + OF^2$. Therefore (Pl. Ge. I. 48) POF is a right angle. In a similar manner it may be shown that PO is perpendicular to any other line through O in the plane ACBD; therefore it is perpendicular to that plane (I. Def. 1.)

Cor. 1.—If a plane be horizontal in any two directions, it is so in every direction.

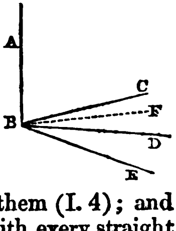
Cor. 2.—The perpendicular PO is less than any oblique line as PB, and therefore it measures the shortest distance from the point P to the plane.

PROPOSITION V. THEOREM.

If three straight lines meet all in one point, and a straight line stand at right angles to each of them in that point, these three straight lines are in one and the same plane.

Let the straight line AB stand at right angles to each of the straight lines BC, BD, BE, in B, the point where they meet; BC, BD, BE, are in one and the same plane.

If not, let, if it be possible, BD and BE be in one plane, and BC be above it; and let a plane pass through AB, BC, the common section of which with the plane, in which BD and BE are, shall be a straight line (I. 3); let this be BF; therefore the three straight lines AB, BC, BF, are all in one plane, namely, that which passes through AB, BC; and because AB stands at right angles to each of the straight lines BD, BE, it is also at right angles to the plane passing through them (I. 4); and therefore makes right angles (I. Def. 1) with every straight line meeting it in that plane; but BF, which is in that plane, meets it; therefore the angle ABF is a right angle; but the



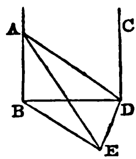
angle ABC, by the hypothesis, is also a right angle; therefore the angle ABF is equal to the angle ABC, and they are both in the same plane; which is impossible: therefore the straight line BC is not above the plane in which are BD and BE. Wherefore the three straight lines BC, BD, BE, are in one and the same plane.

PROPOSITION VI. THEOREM.

If two straight lines be at right angles to the same plane, they shall be parallel to one another.

Let the straight lines AB, CD, be at right angles to the same plane; AB is parallel to CD.

Let them meet the plane in the points B, D, and draw the straight line BD, to which draw DE at right angles, in the same plane; and make DE equal to AB, and join BE, AE, AD. Then, because AB is perpendicular to the plane, it shall make right (I. Def. 1) angles with every straight line which meets it, and is in that plane; but BD, BE, which are in that plane, do each of them meet AB. Therefore each of the angles

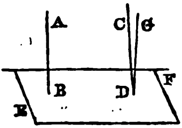


ABD, ABE, is a right angle. For the same reason, each of the angles CDB, CDE, is a right angle; and because AB is equal to DE, and BD common, the two sides AB, BD, are equal to the two ED, DB; and they contain right angles; therefore the base AD is equal (Pl. Ge. I. 4) to the base BE. Again, because AB is equal to DE, and BE to AD, and the base AE common to the triangles ABE, EDA, the angle ABE is equal (Pl. Ge. I. 8) to the angle EDA; but ABE is a right angle; therefore EDA is also a right angle, and ED perpendicular to DA; but it is also perpendicular to each of the two BD, DC; wherefore ED is at right angles to each of the three straight lines BD, DA, DC, in the point in which they meet. Therefore these three straight lines are all in the same plane (I. 5); but AB is in the plane in which are BD, DA, because any three straight lines which meet one another are in one plane (I. 2). Therefore AB, BD, DC, are in one plane; and each of the angles ABD, BDC, is a right angle; therefore AB is parallel to CD.

PROPOSITION VII. THEOREM.

If two straight lines be parallel, and one of them is at right angles to a plane, the other also shall be at right angles to the same plane.

Let AB, CD, be two parallel straight lines, and let one of them AB be at right angles to a plane; the other CD is at right angles to the same plane.



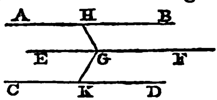
For, if CD be not perpendicular to the plane to which AB is perpendicular, let DG be perpendicular to it. Then (I. 6) DG is parallel to AB; DG and DC therefore are both parallel to AB, and are drawn through the same point D; which is impossible.

PROPOSITION VIII. THEOREM.

Two straight lines which are each of them parallel to the same straight line, though not both in the same plane with it, are parallel to one another.

Let AB, CD, be each of them parallel to EF, and not in the same plane with it; AB shall be parallel to CD.

In EF take any point G, from which draw, in the plane passing through EF, AB, the straight line GH at right angles to EF; and in the plane passing through EF, CD, draw GK at right angles to the same EF. And because EF is perpendicular both to GH and GK, EF is perpendicular to the plane HGK passing through them; and EF is parallel to AB; therefore AB is at right angles (I. 7) to the plane HGK. For the same reason, CD is likewise at right angles to the plane HGK. Therefore AB, CD, are each of them at right angles to the plane HGK. But if two straight lines are at right angles to the same plane, they are parallel (I. 6) to one another. Therefore AB is parallel to CD.

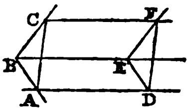


PROPOSITION IX. THEOREM.

If two straight lines meeting one another be parallel to *two others that meet* one another, though not in the same

plane with the first two, the first two and the other two shall contain equal angles.

Let the two straight lines AB, BC, which meet one another, be parallel to the two straight lines DE, EF, that meet one another, and are not in the same plane with AB, BC. The angle ABC is equal to the angle DEF.



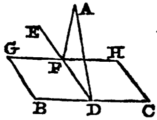
Take BA, BC, ED, EF, all equal to one another; and join AD, CF, BE, AC, DF; because BA is equal and parallel to ED, therefore AD is (Pl. Ge. I. 33) both equal and parallel to BE; for the same reason, CF is equal and parallel to BE; therefore AD and CF are each of them equal and parallel to BE. But straight lines that are parallel to the same straight line, though not in the same plane with it, are parallel (I. 8) to one another; therefore AD is parallel to CF; and it is equal to it, and AC, DF, join them towards the same parts; and therefore AC is equal and parallel to DF; and because AB, BC, are equal to DE, EF, and the base AC to the base DF, the angle ABC is equal (Pl. Ge. I. 8) to the angle DEF.

PROPOSITION X. THEOREM.

To draw a straight line perpendicular to a plane, from a given point above it.

Let A be the given point above the plane BH; it is required to draw from the point A a straight line perpendicular to the plane BH.

In the plane draw any straight line BC, and from the point A draw AD perpendicular to BC. If then AD be also perpendicular to the plane BH, the thing required is already done; but if it be not, from the point D draw, in the plane BH, the straight line DE at right angles to BC; and from the point A draw AF perpendicular to DE; and through F draw GH parallel to BC; and because BC is at right angles to ED and DA, BC is at right angles (I. 4) to the plane passing through ED, DA; and GH is parallel to BC; but if two straight lines be parallel, one of which is at right angles to a plane, the other



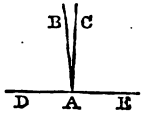
shall be at right (I. 7) angles to the same plane; wherefore GH is at right angles to the plane through ED, DA, and is perpendicular (I. Def. 1) to every straight line meeting it in that plane. But AF, which is in the plane through ED, DA, meets it; therefore GH is perpendicular to AF; and consequently AF is perpendicular to GH; and AF is also perpendicular to DE; therefore AF is perpendicular to each of the straight lines GH, DE. But if a straight line stands at right angles to each of two straight lines in the point of their intersection, it shall also be at right angles to the plane passing through them; and the plane passing through ED, GH, is the plane BH; therefore AF is perpendicular to the plane BH; so that, from the given point A, above the plane BH, the straight line AF is drawn perpendicular to that plane.

Cor.—If it be required from a point C in a plane to erect a perpendicular to that plane, take a point A above the plane, and draw AF perpendicular to the plane; then, if from C a line be drawn parallel to AF, it will be the perpendicular required; for, being parallel to AF, it will be perpendicular to the same plane to which AF is perpendicular.

PROPOSITION XI. THEOREM.

From the same point in a given plane, there cannot be two straight lines at right angles to the plane, upon the same side of it; and there can be but one perpendicular to a plane from a point above the plane.

For, if it be possible, let the two straight lines AC, AB, be at right angles to a given plane from the same point A in the plane, and upon the same side of it; and let a plane pass through BA, AC; the common section of this with the given plane is a straight line passing through A (I. 3). Let DAE be their common section; therefore the straight lines AB, AC, DAE, are in one plane; and because CA is at right angles to the given plane, it shall make right angles with every straight line meeting it in that plane. But DAE, which is in that plane, meets CA; therefore CAE is a right angle. For the same reason, BAE



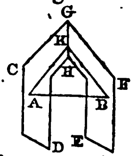
is a right angle; wherefore the angle CAE is equal to the angle BAE; and they are in one plane; which is impossible. Also, from a point above a plane, there can be but one perpendicular to that plane; for if there could be two, they would be parallel (I. 6) to one another; which is absurd.

PROPOSITION XII. THEOREM.

Planes to which the same straight line is perpendicular, are parallel to one another.

Let the straight line AB be perpendicular to each of the planes CD, EF; these planes are parallel to one another.

If not, they shall meet one another when produced; let them meet; their common section shall be a straight line GH, in which take any point K, and join AK, BK; then, because AB is perpendicular to the plane EF, it is perpendicular (I. Def. 1) to the straight line BK which is in that plane; therefore ABK is a right angle; for the same reason, BAK is a right angle; wherefore the two angles ABK, BAK, of the triangle ABK, are equal to two right angles; which is impossible (Pl. Ge. I. 17); therefore the planes CD, EF, though produced, do not meet one another; that is, they are parallel (I. Def. 6.)

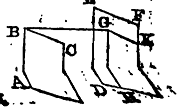


PROPOSITION XIII. THEOREM.

If two straight lines meeting one another be parallel to two straight lines which meet one another, but are not in the same plane with the first two, the plane which passes through these is parallel to the plane passing through the others.

Let AB, BC, two straight lines meeting one another, be parallel to DE, EF, that meet one another, but are not in the same plane with AB, BC; the planes through AB, BC, and DE, EF, shall not meet, though produced.

From the point B draw BG perpendicular (I. 10) to the plane which passes through DE, EF, and let it meet that plane in G; and through G draw GH parallel to ED, and GK parallel to EF. And because BG is perpendicular to the plane through DE, EF, it shall make right angles with every straight line meeting it in that

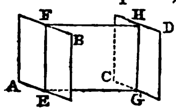


plane (I. Def. 1). But the straight lines GH, GK, in that plane meet it; therefore each of the angles BGH, BGK, is a right angle. And because BA is parallel (I. 8) to GH (for each of them is parallel to DE), the angles GBA, BGH, are together equal (Pl. Ge. I. 29) to two right angles. And BGH is a right angle; therefore also GBA is a right angle, and GB perpendicular to BA. For the same reason, GB is perpendicular to BC. Since therefore the straight line GB stands at right angles to the two straight lines BA, BC, that cut one another in B, GB is perpendicular (I. 4) to the plane through BA, BC; and it is perpendicular to the plane through DE, EF; therefore BG is perpendicular to each of the planes through AB, BC, and DE, EF. But planes to which the same straight line is perpendicular, are parallel (I. 12) to one another. Therefore the plane through AB, BC, is parallel to the plane through DE, EF.

PROPOSITION XIV. THEOREM.

If two parallel planes be cut by another plane, their common sections with it are parallels.

Let the parallel planes AB, CD, be cut by the plane EFHG, and let their common sections with it be EF, GH; EF is parallel to GH.



For the straight lines EF and GH are in the same plane, namely, EFHG, which cuts the planes AB and CD; and they do not meet though produced; for the planes in which they are, do not meet; therefore EF and GH are parallel.

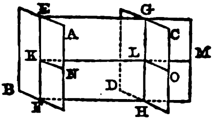
PROPOSITION XV. THEOREM.

If two parallel planes be cut by a third plane, they have the same inclination to that plane.

Let AB and CD be two parallel planes, and EH a third plane cutting them. The planes AB and CD are equally inclined to EH.

Let the straight lines EF and GH be the common sections of the plane EH with the two planes AB and CD; and from K any point in EF, draw in the plane EH the straight line KM at right angles to EF, and let it meet GH in

L; draw also KN at right angles to EF in the plane AB; and through the straight lines KM, KN, let a plane be made to pass cutting the plane CD in the line LO. And because EF and GH are the common sections of the plane EH with the two parallel planes AB and CD, EF is parallel to GH (I. 14). But EF is at right angles to the plane that passes through KN and KM (I. 4), because it is at right angles to the lines KM and KN; therefore GH is also at right angles to the same plane (I. 7), and it is therefore at right angles to the lines LM, LO, which it meets in that plane. Therefore, since LM and LO are at right angles to LG, the common section of the two planes CD and EH, the angle OLM is the inclination of the plane CD to the plane EH (I. Def. 4). For the same reason the angle MKN is the inclination of the plane AB to the plane EH. But because KN and LO are parallel, being the common sections of the parallel planes AB and CD with a third plane, the interior angle NKM is equal to the exterior angle OLM; that is, the inclination of the plane AB to the plane EH, is equal to the inclination of the plane CD to the same plane EH.

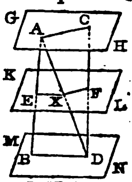


PROPOSITION XVI. THEOREM.

If two straight lines be cut by parallel planes, they shall be cut in the same ratio.

Let the straight lines AB, CD, be cut by the parallel planes GH, KL, MN, in the points A, E, B; C, F, D. As AE is to EB, so is CF to FD.

Join AC, BD, AD, and let AD meet the plane KL in the point X; and join EX, XF. Because the two parallel planes KL, MN, are cut by the plane EBDX, the common sections EX, BD, are parallel (I. 3). For the same reason, because the two parallel planes GH, KL, are cut by the plane AXFC, the common sections AC, XF, are parallel. And because EX is parallel to BD, a side of the triangle ABD, as AE to EB, so is (Pl. Ge. VI. 2) AX to XD. Again, because XF is parallel to AC, a side of the triangle ADC, as AX to XD, so



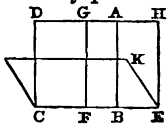
is CF to FD. And it was proved that AX is to XD, as AE to EB. Therefore (Pl. Ge. V. 11), as AE to EB, so is CF to FD.

PROPOSITION XVII. THEOREM.

If a straight line be at right angles to a plane, every plane which passes through it shall be at right angles to that plane.

Let the straight line AB be at right angles to a plane CK; every plane which passes through AB shall be at right angles to the plane CK.

Let any plane DE pass through AB, and let CE be the common section of the planes DE, CK; take any point F in CE, from which draw FG in the plane DE at right angles to CE. And because AB is perpendicular to the plane CK, therefore it is also perpendicular to every straight line meeting it in that plane (I. Def. 1); and consequently it is perpendicular to CE. Wherefore ABF is a right angle; but GFB is likewise a right angle; therefore AB is parallel to FG. And AB is at right angles to the plane CK; therefore FG is also at right angles to the same plane (I. 7). But one plane is at right angles to another plane when the straight lines drawn in one of the planes, at right angles to their common section, are also at right angles to the other plane (I. Def. 2); and any straight line FG in the plane DE, which is at right angles to CE, the common section of the planes, has been proved to be perpendicular to the other plane CK; therefore the plane DE is at right angles to the plane CK. In like manner, it may be proved that all the planes which pass through AB are at right angles to the plane CK.



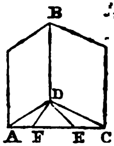
PROPOSITION XVIII. THEOREM.

If two planes cutting one another be each of them perpendicular to a third plane, their common section shall be perpendicular to the same plane.

Let the two planes AB, BC, be each of them perpendicular to a third plane, and let BD be the common section of the first two; BD is perpendicular to the plane ADC.

From D in the plane ADC, draw DE perpendicular to

AD, and DF to DC. Because DE is perpendicular to AD, the common section of the planes AB and ADC; and because the plane AB is at right angles to ADC, DE is at right angles to the plane AB (I. Def. 2), and therefore also to the straight line BD in that plane (I. Def. 1). For the same reason, DF is at right angles to DB. Since BD is therefore at right angles to both the lines DE and DF, it is at right angles to the plane in which DE and DF are, that is, to the plane ADC (I. 4).



SECOND BOOK.

DEFINITIONS.

1. A *solid* is a figure that has length, breadth, and thickness.

2. A *solid angle* is formed by the meeting, in one point, of more than two plane angles not in the same plane.

When the solid angle is contained by three plane angles, it is called a *trihedral* angle; and if it be contained by more than three, it is said to be *polyhedral*. The plane angles are called the *sides* or *faces;* and the common sections of their planes, the *edges* of the angle.

3. A *pyramid* is a solid having a rectilineal figure for its base, and, for its sides, it has triangles that meet in a point without the base, and having for their bases the sides of the base of the pyramid.

The common vertex of the sides is called the *vertex* of the pyramid; and the *altitude* of a pyramid is the perpendicular from its vertex to the plane of its base.

4. A *prism* is a solid contained by plane figures, of which two are opposite, equal, similar, and parallel to one another, and the others are parallelograms.

The parallelograms are called the *sides;* and the other two plane figures the *ends*, one of which is called the *base*. The *altitude* is the perpendicular distance of its two ends. It is said *to be a right* prism when the edges are perpendi-

cular to the base; in other cases it is said to be *oblique*. The surface of the sides of a pyramid or prism is called the *lateral* or *convex surface*. A pyramid or prism is named according to the figure of its base. According as the base is a triangle, a rectangle, a square, or a polygon, it is said to be *triangular*, *rectangular*, *square*, or *polygonal*.

5. A *parallelopiped* is a solid figure contained by six quadrilateral figures, of which every opposite two are parallel.

Any side of a parallelopiped may be called its *base*; and its base, and the side opposite, are called its *ends*. A parallelopiped is just a prism, with a parallelogram for its base; and the definitions of the terms *right*, *oblique*, *altitude*, *lateral* and *convex surface*, are the same as in the case of the prism. A right parallelopiped is said to be *contained* by any three of its edges that belong to one of its trihedral angles, or by any three lines equal to them. If A, B, C, be three of these edges, or any three lines, the parallelopiped contained by them is expressed thus, $A \cdot B \cdot C$.

6. A *cube* is a solid figure contained by six equal squares.

The cube described upon any line, as a line M, is expressed thus, M^3.

7. A *polyhedron* is any solid contained by more than three planes. If it has four sides, it is called a *tetrahedron*; if six, a *hexahedron*; if eight, an *octahedron*; if twelve, a *dodecahedron*; and if twenty, an *icosahedron*.

8. Two polyhedrons are said to be *similar*, when they are contained by the same number of similar faces, similarly situated, and containing equal dihedral angles.

9. A polyhedron is said to be *regular*, when its sides are equal regular polygons of the same kind, and its solid angles equal.

There are only five regular polyhedrons, of 4, 6, 8, 12, and 20 sides respectively, which are named as in the seventh definition. The first is contained by equilateral triangles; the second by squares; the third by equilateral triangles; the fourth by pentagons; and the fifth by equilateral triangles.

10. A *sphere* is a solid described by the revolution of a *semicircle about its* diameter, which remains fixed.

The *axis* of the sphere is the fixed diameter about which the semicircle revolves; and its *centre* is the same as that of the generating semicircle.

11. The *diameter* of a sphere is a straight line passing through the centre of the sphere, and terminated at each extremity by the surface.

12. A *right cone* is a solid described by the revolution of a right-angled triangle about one of the sides containing the right angle, which remains fixed.

The *axis* of the cone is the fixed line about which the generating triangle revolves; and its *base* is the circle described by the revolving side containing the right angle.

13. A *right cylinder* is a solid described by the revolution of a rectangle about one of its sides, which remains fixed.

The *axis* of the cylinder is the fixed line about which the rectangle revolves; and its *bases* or *ends* are the circles described by the opposite revolving sides of the rectangle.

14. *Similar* cones and cylinders are those that have their axes and the diameters of their bases proportional.

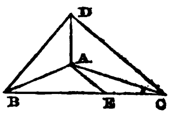
It is evident (Def. 12) that the axis of a cone is the straight line joining its vertex and the centre of its base; and (Def. 13) that the axis of a cylinder is the straight line joining the centres of its two ends.

PROPOSITION I. THEOREM.

Any two of the plane angles that form a trihedral angle, are together greater than the third.

Let the solid angle at A be contained by the three plane angles BAC, CAD, DAB. Any two of them are greater than the third.

If the angles BAC, CAD, DAB, be all equal, it is evident that any two of them are greater than the third. But if they are not, let BAC be that angle which is not less than either of the other two, and is greater than one of them DAB; and at the point A in the straight line AB, make, in the plane which passes



through BA, AC, the angle BAE equal (Pl. Ge. I. 23) to the angle DAB; and make AE equal to AD, and through

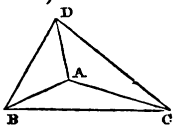
E draw BEC cutting AB, AC, in the points B, C, and join DB, DC. And because DA is equal to AE, and AB is common to the two triangles ABD, ABE, and also the angle DAB equal to the angle EAB; therefore the base DB is equal to the base BE. And because BD, DC, are greater (Pl. Ge. I. 20) than CB, and one of them BD has been proved equal to BE a part of CB, therefore the other DC is greater than the remaining part EC. And because DA is equal to AE, and AC common, but the base DC greater than the base EC; therefore the angle DAC is greater (Pl. Ge. I. 25) than the angle EAC; and, by the construction, the angle DAB is equal to the angle BAE; wherefore the angles DAB, DAC, are together greater than BAE, EAC, that is, than the angle BAC. But BAC is not less than either of the angles DAB, DAC; therefore BAC, with either of them, is greater than the other.

PROPOSITION II. THEOREM.

Every solid angle is contained by plane angles which together are less than four right angles.

First, let the solid angle at A be contained by three plane angles BAC, CAD, DAB. These three together are less than four right angles.

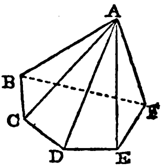
Take in each of the straight lines AB, AC, AD, any points B, C, D, and join BC, CD, DB; then, because the solid angle at B is contained by the three plane angles CBA, ABD, DBC, any two of them are greater (II. 1) than the third; therefore the angles CBA, ABD, are greater than the angle DBC. For the same reason, the angles BCA, ACD, are greater than the angle DCB; and the angles CDA, ADB, greater than BDC; wherefore the six angles CBA, ABD, BCA, ACD, CDA, ADB, are greater than the three angles DBC, BCD, CDB; but the three angles DBC, BCD, CDB, are equal to two right angles (Pl. Ge. I. 32); therefore the six angles CBA, ABD, BCA, ACD, CDA, ADB, are greater than two right angles; and because the three angles of each of the triangles ABC, ACD, ADB, are equal to two right angles, therefore the nine angles of these three triangles,



namely, the angles CBA, BAC, ACB, ACD, CDA, DAC, ADB, DBA, BAD, are equal to six right angles. Of these, the six angles CBA, ACB, ACD, CDA, ADB, DBA, are greater than two right angles; therefore the remaining three angles BAC, DAC, BAD, which contain the solid angle at A, are less than four right angles.

Next, let the solid angle at A be contained by any number of plane angles BAC, CAD, DAE, EAF, FAB; these together are less than four right angles.

Let the planes in which the angles are, be cut by a plane, and let the common sections of it with those planes be BC, CD, DE, EF, FB; and because the solid angle at B is contained by three plane angles CBA, ABF, FBC, of which any two are greater (II. 1) than the third, the angles CBA, ABF, are greater than the angle FBC. For the same reason, the two plane angles at each of the points C, D, E, F, namely, the angles which are at the bases of the triangles having the common vertex A, are greater than the third angle at the same point, which is one of the angles of the polygon BCDEF; therefore all the angles at the bases of the triangles are together greater than all the angles of the polygon; and because all the angles of the triangles are together equal to twice as many right angles as there are triangles (Pl. Ge. I. 32); that is, as there are sides in the polygon BCDEF; and because all the angles of the polygon, together with four right angles, are likewise equal to twice as many right angles as there are sides in the polygon (Pl. Ge. I. 32, Cor. 1); therefore all the angles of the triangles are equal to all the angles of the polygon, together with four right angles. But all the angles at the bases of the triangles are greater than all the angles of the polygon, as has been proved. Wherefore, the remaining angles of the triangles, namely, those at the vertex, which contain the solid angle at A, are less than four right angles.



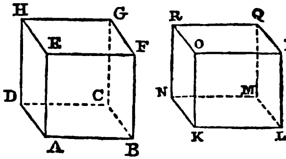
PROPOSITION III. THEOREM.

If two solids be contained by the same number of equal and similar *planes*, similarly situated, and if the inclination

of any two contiguous planes in the one solid be the same with the inclination of the two equal, and similarly situated planes in the other, the solids themselves are equal and similar.

Let AG and KQ be two solids contained by the same number of equal and similar planes, similarly situated, so that the plane AC is similar and equal to the plane KM, the plane AF to the plane KP, BG to LQ, GD to QN, DE to NO, and FH to PR. Let also the inclination of the plane AF to the plane AC be the same with that of the plane KP to the plane KM, and so of the rest; the solid KQ is equal and similar to the solid AG.

Let the solid KQ be applied to the solid AG, so that the bases KM and AC, which are equal and similar, may coincide, the point N coinciding with the point D, K with A, L with B, and so on. And because the plane KM coincides with the plane AC, and, by hypothesis, the inclination of KR to KM is the same



with the inclination of AH to AC, the plane KR will be upon the plane AH, and will coincide with it, because they are similar and equal, and because their equal sides KN and AD coincide. And in the same manner, it is shown that the other planes of the solid KQ coincide with the other planes of the solid AG, each with each; wherefore the solids KQ and AG do wholly coincide, and are equal and similar to one another.

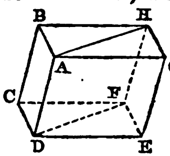
PROPOSITION IV. THEOREM.

If a solid be contained by six planes, two and two of which are parallel, the opposite planes are similar and equal parallelograms.

Let the solid CDGH be contained by the parallel planes AC, GF; BG, CE; FB, AE; its opposite planes are similar and equal parallelograms.

Because the two parallel planes BG, CE, are cut by the plane AC, their common sections AB, CD, are parallel

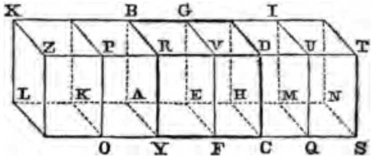
(I. 14). Again, because the two parallel planes BF, AE, are cut by the plane AC, their common sections AD, BC, are parallel; and AB is parallel to CD; therefore AC is a parallelogram. In like manner, it may be proved that each of the figures CE, FG, GB, BF, AE, is a parallelogram. Join AH, DF; and because AB is parallel to DC, and BH to CF; the two straight lines AB, BH, which meet one another, are parallel to DC and CF, which meet one another; wherefore, though the first two are not in the same plane with the other two, they contain equal angles (I. 9); the angle ABH is therefore equal to the angle DCF. And because AB, BH, are equal to DC, CF, and the angle ABH equal to the angle DCF; therefore the base AH is equal to the base DF, and the triangle ABH to the triangle DCF. For the same reason, the triangle AGH is equal to the triangle DEF; and therefore the parallelogram BG is equal and similar to the parallelogram CE. In the same manner, it may be proved that the parallelogram AC is equal and similar to the parallelogram GF, and the parallelogram AE to BF.



PROPOSITION V. THEOREM.

If a solid parallelopiped be cut by a plane parallel to two of its opposite planes, it will be divided into two solids, which will be to one another as their bases.

Let the solid parallelopiped ABCD be cut by the plane EV, which is parallel to the opposite planes AR, HD, and divides the whole into the solids ABFV, EGCD; as the base AEFY to the base EHCF, so is the solid ABFV to the solid EGCD.



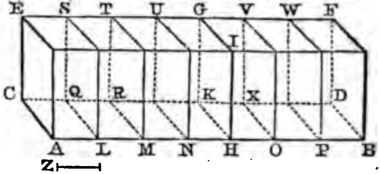
Produce AH both ways, and take any number of straight lines HM, MN, each equal to EH, and any number AK, KL, each equal to EA, and complete the parallelograms LO, KY, HQ, MS, and the solids LP, KR, HU, MT. Then, because

the straight lines LK, KA, AE, are all equal, and also the straight lines KO, AY, EF, which make equal angles with LK, KA, AE, the parallelograms LO, KY, AF, are equal and similar (Pl. Ge. VI. 20); and likewise the parallelograms KX, KB, AG; as also (II. 4) the parallelograms LZ, KP, AR, because they are opposite planes. For the same reason, the parallelograms EC, HQ, MS, are equal; and the parallelograms HG, HI, IN, as also HD, MU, NT; therefore three planes of the solid LP are equal and similar to three planes of the solid KR, as also to three planes of the solid AV; but the three planes opposite to these three are equal and similar to them in the several solids; therefore the solids LP, KR, AV, are contained by equal and similar planes. And because the planes LZ, KP, AR, are parallel, and are cut by the plane XV, the inclination of LZ to XP is equal to that of KP to PB, or of AR to BV (I. 15); and the same is true of the other contiguous planes; therefore the solids LP, KR, and AV, are equal to one another (II. 3). For the same reason, the three solids ED, HU, MT, are equal to one another; therefore what multiple soever the base LF is of the base AF, the same multiple is the solid LV of the solid AV; for the same reason, whatever multiple the base NF is of the base HF, the same multiple is the solid NV of the solid ED; and if the base LF be equal to the base NF, the solid LV is equal to the solid NV; and if the base LF be greater than the base NF, the solid LV is greater than the solid NV; and if less, less. Since then there are four magnitudes, namely, the two bases AF, FH, and the two solids AV, ED, and of the base AF and solid AV, the base LF and solid LV are any equimultiples whatever; and of the base FH and solid ED, the base FN and solid NV are any equimultiples whatever; and it has been proved, that if the base LF is greater than the base FN, the solid LV is greater than the solid NV; and if equal, equal; and if less, less. Therefore (Pl. Ge. V. Def. 10), as the base AF is to the base FH, so is the solid AV to the solid ED.

Scholium.—This proposition may be demonstrated by the principle in the twenty-seventh proposition of the additional *Fifth Book*, thus :—

Let the parallelopiped AF be cut by a plane GH parallel to either of its sides AE or BF, then AG : GB = AK : KB.

For, let the bases AK, KB, be commensurable, and hence the sides AH, HB, are so too. Let AH and HB contain a common measure Z 4 and 3 times respectively. Divide AH into 4 and HB into 3 equal parts in L, M, N, O, and P, and through these points let planes LS, MT, &c. pass, parallel to AE, then AG will be divided into 4, and HF into 3, equal parallelopipeds, AS, LT, &c. The figures AS, LT, &c. are parallelopipeds, for their opposite sides are parallel (II. Def. 5), and hence the opposite sides are equal and similar parallelograms (II. 4). Also the parallelograms AQ, LR, are equal, for AL = LM, and AM is parallel to CR. For a similar reason EQ = SR; and also AE = LS. The parallelograms opposite to these are also equal (II. 4); therefore the two parallelopipeds AS, LT, are contained by the same number of equal and similar parallelograms, similarly situated. The parallel planes AE, LS, are cut by the plane EK; therefore the inclination of AE and EQ is equal to that of LS and SR; and the same may be proved of the inclinations of the other sides of AS and LT. Hence (II. 3) AS = LT. It may be similarly proved that the parallelopipeds LT, MU, NG, HV, OW, and PF, are equal. Hence AG : GB = 4 : 3; but AK : KB = 4 : 3; therefore AG : GB = AK : KB. The same proportion is similarly proved, whatever be the number of times that AH : HB contain their common measure, when commensurable; hence the proportion exists when they are incommensurable (Pl. Ge. Ad. V. 27).

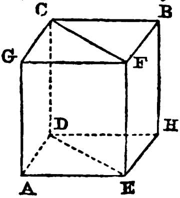


COR.—Because the parallelogram AF (former figure) is to the parallelogram FH as YF to FC (Pl. Ge. VI. 1), therefore the solid AV is to the solid ED as YF to FC.

PROPOSITION VI. THEOREM.

If a solid parallelopiped be cut by a plane passing through the diagonals of two of the opposite planes, it shall be cut in two equal prisms.

Let AB be a solid parallelopiped, and DE, CF, the diagonals of the opposite parallelograms AH, GB, namely, those which are drawn betwixt the equal angles in each; and because CD, FE, are each of them parallel to GA, though not in the same plane with it, CD, FE, are parallel (I. 8); wherefore the diagonals CF, DE, are in the plane in which the parallels are, and are themselves parallels (I. 14); and the plane CDEF shall cut the solid AB into two equal parts.



Because the triangle CGF is equal (Pl. Ge. I. 34) to the triangle CBF, and the triangle DAE to DHE; and that the parallelogram CA is equal (II. 4) and similar to the opposite one BE; and the parallelogram GE to CH; therefore the planes which contain the prisms CAE, CBE, are equal and similar, each to each; and they are also equally inclined to one another, because the planes AC, EB, are parallel, as also AF and BD, and they are cut by the plane CE; therefore the prism CAE is equal to the prism CBE (II. 3), and the solid AB is cut into two equal prisms by the plane CDEF.

Def.—The *insisting* straight lines of a parallelopiped, mentioned in the following propositions, are the sides of the parallelograms betwixt the base and the plane parallel to it.

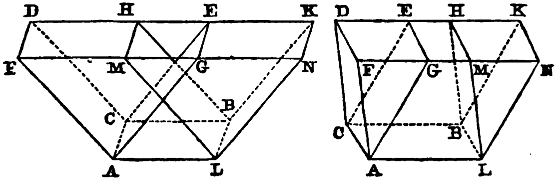
PROPOSITION VII. THEOREM.

Solid parallelopipeds upon the same base, and of the same altitude, the insisting straight lines of which are terminated in the same straight lines in the plane opposite to the base, are equal to one another.

Let the solid parallelopipeds AH, AK, be upon the same base AB, and of the same altitude, and let their insisting straight lines AF, AG, LM, LN, be terminated in the same straight line FN; and CD, CE, BH, BK, be terminated in the same straight line DK; the solid AH is equal to the solid AK.

Because CH, CK, are parallelograms, CB is equal (Pl. Ge. I. 34) to each of the opposite sides DH, EK; wherefore DH is equal to EK. Add, or take away the common

part HE; then DE is equal to HK; wherefore also the triangle CDE is equal (Pl. Ge. I. 38) to the triangle BHK;



and the parallelogram DG is equal (Pl. Ge. I. 36) to the parallelogram HN. For the same reason, the triangle AFG is equal to the triangle LMN, and the parallelogram CF is equal (II. 4) to the parallelogram BM, and CG to BN; for they are opposite. Therefore, the planes which contain the prism DAG are similar and equal to those which contain the prism HLN, each to each; and the contiguous planes are also equally inclined to one another (I. 15), because that the parallel planes AD and LH, as also AE and LK, are cut by the same plane DN; therefore the prisms DAG, HLN, are equal (II. 3). If therefore the prism LNH be taken from the solid, of which the base is the parallelogram AB, and FDKN the plane opposite to the base; and if from this same solid there be taken the prism AGD, the remaining solid, namely, the parallelopiped AH, is equal to the remaining parallelopiped AK.

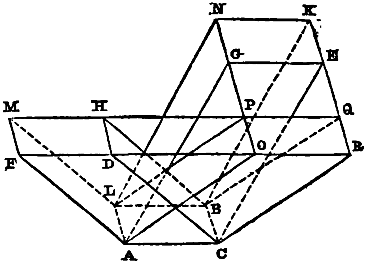
PROPOSITION VIII. THEOREM.

Solid parallelopipeds upon the same base, and of the same altitude, the insisting straight lines of which are not terminated in the same straight lines in the plane opposite to the base, are equal to one another.

Let the parallelopipeds CM, CN, be upon the same base AB, and of the same altitude, but their insisting straight lines AF, AG, LM, LN, CD, CE, BH, BK, not terminated in the same straight lines; the solids CM, CN, are equal to one another.

Produce FD, MH, and NG, KE, and let them meet one another in the points O, P, Q, R; and join AO, LP, BQ, CR. *Because* the planes (II. Def. 5) LBHM and ACDE

are parallel, and because the plane LBHM is that in which are the parallels LB, MHPQ, and in which also is the figure BLPQ; and because the plane ACDF is that in which are the parallels AC, FDOR, and in which also is the figure CAOR; therefore the figures BLPQ, CAOR, are in parallel planes. In like manner, because the planes ALNG and CBKE are parallel, and the plane ALNG is that in which are the parallels AL, OPGN, and in which also is the figure ALPO; and the plane CBKE is that in which are the parallels CB, RQEK, and in which also is the figure CBQR; therefore the figures ALPO, CBQR, are in parallel planes. But the planes ACBL, ORQP, are also parallel; therefore the solid CP is a parallelopiped. Now the solid parallelopiped CM is equal to the solid parallelopiped CP; because they are upon the same base, and their insisting straight lines AF, AO, CD, CR, LM, LP, BH, BQ, are in the same straight lines FR, MQ; and the solid CP is equal to the solid CN; for they are upon the same base ACBL, and their insisting straight lines AO, AG, LP, LN, CR, CE, BQ, BK, are in the same straight lines ON, RK; therefore the solid CM is equal to the solid CN.



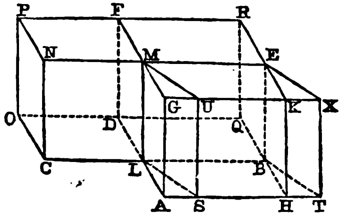
PROPOSITION IX. THEOREM.

Solid parallelopipeds which are upon equal bases, and of the same altitude, are equal to one another.

Let the solid parallelopipeds AE, CF, be upon equal bases AB, CD, and be of the same altitude; the solid AE is equal to the solid CF.

Case 1. Let the insisting straight lines be at right angles to the bases AB, CD, and let the bases be placed in the same plane, and so as that the sides CL, LB, be in a straight *line; therefore the* straight line LM, which is at right angles

to the plane in which the bases are, in the point L, is common (I. 11) to the two solids AE, CF; let the other insisting lines of the solids be AG, HK, BE, DF, OP, CN; and first, let the angle ALB be equal to the angle CLD; then AL, LD, are in a straight line (Pl. Ge. I. 14). Produce OD, HB, and let them meet in Q, and complete

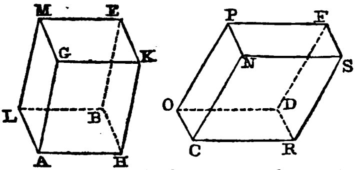


the solid parallelopiped LR, the base of which is the parallelogram LQ, and of which LM is one of its insisting straight lines; therefore, because the parallelogram AB is equal to CD, as the base AB is to the base LQ, so is (Pl. Ge. V. 7) the base CD to the same LQ; and because the solid parallelopiped AR is cut by the plane LMEB, which is parallel to the opposite planes AK, DR; as the base AB is to the base LQ, so is (II. 5) the solid AE to the solid LR. For the same reason, because the solid parallelopiped CR is cut by the plane LMFD, which is parallel to the opposite planes CP, BR; as the base CD to the base LQ, so is the solid CF to the solid LR; but as the base AB to the base LQ, so the base CD to the base LQ, as has been proved; therefore as the solid AE to the solid LR, so is the solid CF to the solid LR; and therefore the solid AE is equal (Pl. Ge. V. 9) to the solid CF.

But let the solid parallelopipeds SE, CF, be upon equal bases SB, CD, and be of the same altitude, and let their insisting straight lines be at right angles to the bases; and place the bases SB, CD, in the same plane, so that CL, LB, be in a straight line; and let the angles SLB, CLD, be unequal; the solid SE is also in this case equal to the solid CF. Produce DL, TS, until they meet in A, and from B draw BH parallel to DA; and let HB, OD, produced, meet in Q, and complete the solids AE, LR; therefore the solid AE, of which the base is the parallelogram LE, and AK the plane opposite to it, is equal (II. 7) to the solid SE, of which the base is LE, and SX the plane opposite; for they are upon *the same* base LE, and of the same altitude, and

their insisting straight lines, namely, LA, LS, BH, BT, MG, MU, EK, EX, are in the same straight lines AT, GX; and because the parallelogram AB is equal to SB, for they are upon the same base LB, and between the same parallels LB, AT; and because the base SB is equal to the base CD; therefore the base AB is equal to the base CD; but the angle ALB is equal to the angle CLD; therefore, by the first case, the solid AE is equal to the solid CF; but the solid AE is equal to the solid SE, as was demonstrated; therefore the solid SE is equal to the solid CF.

Case 2. If the insisting straight lines AG, HK, BE, LM, CN, RS, DF, OP, be not at right angles to the bases AB, CD; in this case likewise the solid AE is equal to the solid CF. Because solid parallelopipeds on the same base, and of the same altitude, are equal (II. 8), if two solid parallelopipeds be constituted on the bases AB and CD of the same altitude with the solids AE and CF, and with



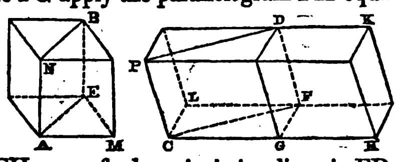
their insisting lines perpendicular to their bases, they will be equal to the solids AE and CF; and, by the first case of this proposition, they will be equal to one another; wherefore, the solids AE and CF are also equal.

PROPOSITION X. THEOREM.

Solid parallelopipeds which have the same altitude, are to one another as their bases.

Let AB, CD, be solid parallelopipeds of the same altitude; they are to one another as their bases; that is, as the base AE to the base CF, so is the solid AB to the solid CD.

To the straight line FG apply the parallelogram FH equal (Pl. Ge. I. 45, Cor.) to AE, so that the angle FGH be equal to the angle LCG; and complete the solid parallelopiped



GK upon the base FH, one of whose insisting lines is FD,

whereby the solids CD, GK, must be of the same altitude; therefore the solid AB is equal (II. 9) to the solid GK, because they are upon equal bases AE, FH, and are of the same altitude; and because the solid parallelopiped CK is cut by the plane DG, which is parallel to its opposite planes, the base HF is (II. 5) to the base FC, as the solid HD to the solid DC. But the base HF is equal to the base AE, and the solid GK to the solid AB; therefore, as the base AE to the base CF, so is the solid AB to the solid CD.

COR. 1.—From this it is manifest that prisms upon triangular bases, of the same altitude, are to one another as their bases. Let the prisms BNM, DPG, the bases of which are the triangles AEM, CFG, have the same altitude; complete the parallelograms AE, CF, and the solid parallelopipeds AB, CD, in the first of which let AN, and in the other let CP, be one of the insisting lines. And because the solid parallelopipeds AB, CD, have the same altitude, they are to one another as the base AE is to the base CF; wherefore the prisms, which are their halves (II. 6) are to one another, as the base AE to the base CF; that is, as the triangle AEM to the triangle CFG.

COR. 2.—Also a prism and a parallelopiped, which have the same altitude, are to one another as their bases; that is, the prism BNM is to the parallelopiped CD as the triangle AEM to the parallelogram LG. For, by the last Cor., the prism BNM is to the prism DPG as the triangle AME to the triangle CGF, and therefore the prism BNM is to twice the prism DPG as the triangle AME to twice the triangle CGF (Pl. Ge. V. 4); that is, the prism BNM is to the parallelopiped CD as the triangle AME to the parallelogram LG.

PROPOSITION XI. THEOREM.

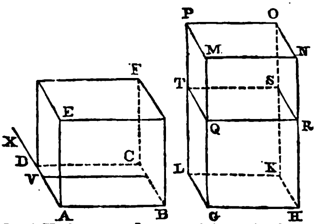
Solid parallelopipeds are to one another in the ratio that is compounded of the ratios of their bases, and of their altitudes.

Let AF and GO be two solid parallelopipeds, of which the bases are the parallelograms AC and GK, and the altitudes, the *perpendiculars* let fall on the planes of these bases

from any point in the opposite planes EF and MO; the solid AF is to the solid GO in a ratio compounded of the ratios of the base AC to the base GK, and of the perpendicular on AC to the perpendicular on GK.

Case 1. When the insisting lines are perpendicular to the bases AC and GK, or when the solids are upright.

In GM, one of the insisting lines of the solid GO, take GQ equal to AE, one of the insisting lines of the solid AF, and through Q let a plane pass parallel to the plane GK, meeting the other insisting lines of the solid GO in the points R, S, and T. It is evident that GS is a solid parallelopiped (II. 4), and that it has the same altitude with AF, namely,



GQ or AE. Now, the solid AF is to the solid GO in a ratio compounded of the ratios of the solid AF to the solid GS (Pl. Ge. V. Def. 17), and of the solid GS to the solid GO; but the ratio of the solid AF to the solid GS, is the same with that of the base AC to the base GK (II. 10), because their altitudes AE and GQ are equal; and the ratio of the solid GS to the solid GO, is the same with that of GQ to GM, for they are as their bases GT, GP (II. 5), which are as GQ to GM; therefore, the ratio which is compounded of the ratios of the solid AF to the solid GS, and of the solid GS to the solid GO, is the same with the ratio which is compounded of the ratios of the base AC to the base GK, and of the altitude AE to the altitude GM (Pl. Ge. V. E). But the ratio of the solid AF to the solid GO is that which is compounded of the ratios of AF to GS, and of GS to GO; therefore, the ratio of the solid AF to the solid GO is compounded of the ratios of the base AC to the base GK, and of the altitude AE to the altitude GM.

Case 2. When the insisting lines are not perpendicular to the bases.

Let the parallelograms AC and GK be the bases as before, and let AE and GM be the altitudes of two parallelopipeds *Y* and *Z* on these bases. Then, if the upright

parallelopipeds AF and GO be constituted on the bases AC and GK, with the altitudes AE and GM, they will be equal to the parallelopipeds Y and Z (II. 9). Now, the solids AF and GO, by the first case, are in the ratio compounded of the ratios of the bases AC and GK, and of the altitudes AE and GM; therefore, also, the solids Y and Z have to one another a ratio that is compounded of these same ratios.

COR. 1.—Hence, two straight lines may be found having the same ratio with the two parallelopipeds AF and GO. To AB, one of the sides of the parallelogram AC, apply the parallelogram BV equal to GK, having an angle equal to the angle BAD (Pl. Ge. I. 44); and as AE to GM, so let AV be to AX (Pl. Ge. VI. 12), then AD is to AX as the solid AF to the solid GO. For the ratio of AD to AX is compounded of the ratios (Pl. Ge. V. Def. 17) of AD to AV, and of AV to AX; but the ratio of AD to AV is the same with that of the parallelogram AC to the parallelogram BV or GK; and the ratio of AV to AX is the same with that of AE to GM; therefore, the ratio of AD to AX is compounded of the ratios of AC to GK, and of AE to GM (Pl. Ge. V. E). But the ratio of the solid AF to the solid GO is compounded of the same ratios; therefore, as AD to AX, so is the solid AF to the solid GO.

COR. 2.—Hence all prisms are to one another in the ratio compounded of the ratios of their bases, and of their altitudes. For every prism is equal to a parallelopiped of the same altitude with it, and of an equal base (II. 10, Cor. 2).

COR. 3.—The right rectangular parallelopipeds contained by the corresponding lines of three analogies, are proportional.

Let $A : B = C : D$, $E : F = G : H$, and $I : K = L : M$, be three analogies, the terms of which are lines, then $A \cdot E \cdot I : B \cdot F \cdot K = C \cdot G \cdot L : D \cdot H \cdot M$.

For let P, Q, R, and S, denote these parallelopipeds; I, K, L, and M, their respective altitudes; and consequently the rectangles $A \cdot E$, $B \cdot F$, $C \cdot G$, and $D \cdot H$, their bases. Then (*Pl. Ge. VI. 23, Cor. 1*) $A \cdot E : B \cdot F = C \cdot G : D \cdot H$. But

(II. 11) $P : Q = (A \cdot E : B \cdot F, I : K)$, and $R : S = (C \cdot G : D \cdot H, L : M)$; and the two component ratios of $P : Q$ are equal respectively to those of $R : S$; therefore (Pl. Ge. V. E.) $P : Q = R : S$.

COR. 4.—A right rectangular parallelopiped is equal to the cube described on any unit of measure, multiplied by the product of the numbers denoting the number of times that it is contained in any three contiguous edges of the parallelopiped, or its length, breadth, and height.

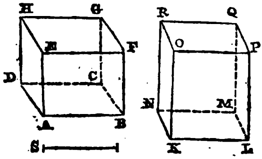
Let A, B, and C, be the three sides; and a, b, and c, the three numbers, whether terminate or interminate, that denote the number of times that a line M, taken as the unit of measure, is contained in these sides respectively; then, considering C and M as the respective altitudes of the parallelopiped $A \cdot B \cdot C$ and the cube M^3, and $A \cdot B$ and M^2 as their bases (II. 11), $A \cdot B \cdot C : M^3 = (A \cdot B : M^2, C : M)$. But (Pl. Ge. Qu. 10, and Ad. V. 7) $(A \cdot B : M^2, C : M) = (ab : 1, c : 1) = abc : 1$ (Pl. Ge. Ad. V. 26). Therefore $A \cdot B \cdot C : M^3 = abc : 1$, or $A \cdot B \cdot C = abc\ M^3$.

PROPOSITION XII. THEOREM.

Solid parallelopipeds which have their bases and altitudes reciprocally proportional, are equal; and parallelopipeds which are equal, have their bases and altitudes reciprocally proportional.

Let AG and KQ be two solid parallelopipeds, of which the bases are AC and KM, and the altitudes AE and KO, and let AC be to KM as KO to AE, the solids AG and KQ are equal.

As the base AC to the base KM, so let the straight line KO be to the straight line S. Then, since AC is to KM as KO to S, and also by hypothesis, AC to KM as KO to AE, KO has the same ratio to S that it has to AE (Pl. Ge. V. 11); wherefore AE is equal to S. But the solid AG is to the solid KQ in the ratio compounded of the ratios of AE to KQ, and of AC to KM (II. 11), that is, in the ratio compounded of the ratios of AE to KO, and of



KO to S. And the ratio of AE to S is also compounded of the same ratios (Pl. Ge. V. Def. 17); therefore, the solid AG has to the solid KQ the same ratio that AE has to S. But AE was proved to be equal to S, therefore AG is equal to KQ.

Again, if the solids AG and KQ be equal, the base AC is to the base KM as the altitude KO to the altitude AE. Take S, so that AC may be to KM as KO to S, and it will be shown, as was done above, that the solid AG is to the solid KQ as AE to S. Now, the solid AG is, by hypothesis, equal to the solid KQ; therefore, AE is equal to S; but, by construction, AC is to KM as KO is to S; therefore, AC is to KM as KO to AE.

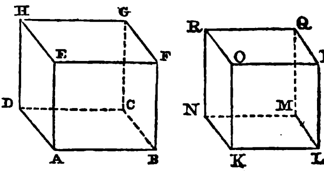
COR.—In the same manner, may it be demonstrated that equal prisms have their bases and altitudes reciprocally proportional, and conversely.

PROPOSITION XIII. THEOREM.

Similar solid parallelopipeds are to one another in the triplicate ratio of their homologous sides.

Let AG, KQ, be two similar parallelopipeds, of which AB and KL are two homologous sides; the ratio of the solid AG to the solid KQ is triplicate of the ratio of AB to KL.

Because the solids are similar, the parallelograms AF, KP, are similar (II. Def. 8), as also the parallelograms AH, KR; therefore, the ratios of AB to KL, of AE to KO, and of AD to KN, are all equal (Pl. Ge. VI. Def. 9). But the ratio of the solid AG to the solid KQ is compounded of the ratios



of AC to KM, and of AE to KO. Now, the ratio of AC to KM, because they are equiangular parallelograms, is compounded (Pl. Ge. VI. 23) of the ratios of AB to KL, and of AD to KN. Wherefore, the ratio of AG to KQ is compounded of the three ratios of AB to KL, AD to KN, and AE to KO; and these three ratios have already been proved to be equal; therefore, the ratio that is compounded

of them, namely, the ratio of the solid AG to the solid KQ, is triplicate of any of them (Pl. Ge. V. Def. 19); it is therefore triplicate of the ratio of AB to KL.

Cor. 1.—If as AB to KL, so KL to m, and as KL to m, so is m to n, then AB is to n as the solid AG to the solid KQ. For the ratio of AB to n is triplicate of the ratio of AB to KL, and is therefore equal to that of the solid AG to the solid KQ.

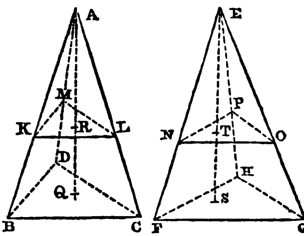
Cor. 2.—As cubes are similar solids, therefore the cube on AB is to the cube on KL in the triplicate ratio of AB to KL, that is, in the same ratio with the solid AG to the solid KQ. Similar solid parallelopipeds are therefore to one another as the cubes on their homologous sides.

Cor. 3.—In the same manner, it is proved that similar prisms are to one another in the triplicate ratio, or in the ratio of the cubes, of their homologous sides.

PROPOSITION XIV. THEOREM.

If two triangular pyramids which have equal bases and altitudes be cut by planes that are parallel to the bases, and at equal distances from them, the sections are equal to one another.

Let ABCD and EFGH be two pyramids, having equal bases BDC and FGH, and equal altitudes, namely, the perpendiculars AQ and ES drawn from A and E upon the planes BDC and FGH; and let them be cut by planes parallel to BDC and FGH, and at equal altitudes QR and ST above those planes, and let the sections be the triangles KLM, NOP; KLM and NOP are equal to one another.



Because the plane ABD cuts the parallel planes BDC, KLM, the common sections BD and KM are parallel (I. 14). For the same reason, *DC and ML* are parallel. Since therefore KM and ML *are parallel to BD* and DC, each to each, though not in the

same plane with them, the angle KML is equal to the angle BDC (I. 9). In like manner, the other angles of these triangles are proved to be equal; therefore the triangles are equiangular, and consequently similar; and the same is true of the triangles NOP, FGH.

Now, since the straight lines ARQ, AKB, meet the parallel planes BDC and KML, they are cut by them proportionally (I. 16), or QR : RA = BK : KA; and AQ : AR = AB : AK (Pl. Ge. V. 18), for the same reason, ES : ET = EF : EN; therefore AB : AK = EF : EN, because AQ is equal to ES, and AR to ET. Again, because the triangles ABC, AKL, are similar,

AB : AK = BC : KL; and for the same reason,
EF : EN = FG : NO; therefore
BC : KL = FG : NO. And, when four straight lines are proportionals, the similar figures described on them are also proportionals (Pl. Ge. VI. 22); therefore the triangle BCD is to the triangle KLM as the triangle FGH to the triangle NOP; but the triangles BDC, FGH, are equal; therefore the triangle KLM is also equal to the triangle NOP (Pl. Ge. V. 14).

Cor. 1.—Because it has been shown that the triangle KLM is similar to the base BCD, therefore, any section of a triangular pyramid parallel to the base, is a triangle similar to the base. And in the same manner, it is shown that the sections parallel to the base of a polygonal pyramid are similar to the base.

Cor. 2.—Hence also, the sections parallel to the bases of two polygonal pyramids, and at equal distances from the bases, are equal to one another.

PROPOSITION XV. THEOREM.

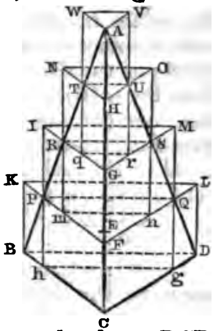
A series of prisms of the same altitude may be circumscribed about any pyramid, such that the sum of the prisms shall exceed the pyramid by a solid less than any given solid.

Let ABCD be a pyramid, and Z* a given solid; a series of prisms having all the same altitude, may be circumscribed

* The solid Z is not represented in the figure of this or the following *proposition.*

about the pyramid ABCD, so that their sum shall exceed ABCD by a solid less than Z.

Let Z be equal to a prism standing on the same base with the pyramid, namely, the triangle BCD, and having for its altitude the perpendicular drawn from a certain point E in the line AC upon the plane BCD. It is evident that CE, multiplied by a certain number *m*, will be greater than AC; divide CA into as many equal parts as there are units in *m*, and let these be CF, FG, GH, HA, each of which will be less than CE. Through each of the points F, G, H, let planes be made to pass parallel to the plane BCD, making with the sides of the pyramid the section FPQ, GRS, HTU, which will



be all similar to one another, and to the base BCD (II. 14, Cor. 1). From the point B draw in the plane of the triangle ABC the straight line BK parallel to CF, meeting FP produced in K. In like manner, from D draw DL parallel to CF, meeting FQ in L. Join KL, and it is plain that the solid KBCDLF is a prism (II. Def. 4). By the same construction, let the prisms PM, RO, TV, be described. Also, let the straight line IP, which is in the plane of the triangle ABC, be produced till it meet BC in h; and let MQ be produced till it meet DC in g. Join hg; then hCgQFP is a prism, and is equal to the prism PM (II. 10, Cor. 1). In the same manner is described the prism mS equal to the prism RO, and the prism qU equal to the prism TV. The sum, therefore, of all the inscribed prisms hQ, mS, and qU, is equal to the sum of the prisms PM, RO, and TV, that is, to the sum of all the circumscribed prisms except the prism BL; wherefore, BL is the excess of the prisms circumscribed about the pyramid ABCD above the prisms inscribed within it. But the prism BL is less than the prism which has the triangle BCD for its base, and for its altitude the perpendicular from E upon the plane BCD; and the prism which has BCD for its base, and the perpendicular from E for its altitude, is by

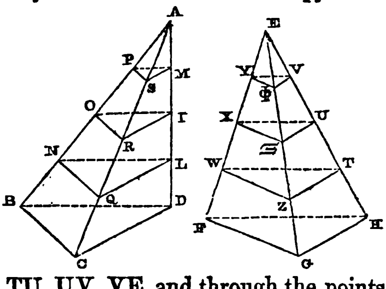
hypothesis equal to the given solid Z; therefore, the excess of the circumscribed, above the inscribed prisms, is less than the given solid Z. But the excess of the circumscribed prisms above the inscribed, is greater than their excess above the pyramid ABCD, because ABCD is greater than the sum of the inscribed prisms. Much more, therefore, is the excess of the circumscribed prisms above the pyramid, less than the solid Z. A series of prisms of the same altitude has therefore been circumscribed about the pyramid ABCD exceeding it by a solid less than the given solid Z.

PROPOSITION XVI. THEOREM.

Pyramids that have equal bases and altitudes are equal to one another.

Let ABCD, EFGH, be two pyramids that have equal bases BCD, FGH, and also equal altitudes, namely, the perpendiculars drawn from the vertices A and E upon the planes BCD, FGH. The pyramid ABCD is equal to the pyramid EFGH.

If they are not equal, let the pyramid EFGH exceed the pyramid ABCD by the solid Z. Then, a series of prisms of the same altitude may be described about the pyramid ABCD that shall exceed it, by a solid less than Z (II. 15); let these be the prisms that have for their bases the triangles BCD, NQL, ORI, PSM. Divide EH into the same number of equal parts into which AD is divided, namely, HT, TU, UV, VE, and through the points T, U, and V, let the sections TZW, UΞX, VΦY, be made parallel to the base FGH. The section NQL is equal to the section WZT (II. 14); as also ORI to XΞU, and PSM to YΦV; and therefore, also, the prisms that stand upon the equal sections are equal (II. 10, Cor. 1), that is, the *prism which stands* on the base BCD, and which is between



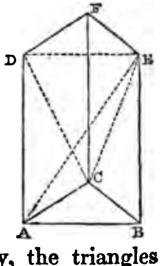
the planes BCD and NQL, is equal to the prism which stands on the base FGH, and which is between the planes FGH and WZT; and so of the rest, because they have the same altitude; wherefore, the sum of all the prisms described about the pyramid ABCD is equal to the sum of all those described about the pyramid EFGH. But the excess of the prisms described about the pyramid ABCD above the pyramid ABCD, is less than Z; and therefore the excess of the prisms described about the pyramid EFGH above the pyramid ABCD, is also less than Z. But the excess of the pyramid EFGH above the pyramid ABCD, is equal to Z, by hypothesis; therefore, the pyramid EFGH exceeds the pyramid ABCD, more than the prisms described about EFGH exceed the same pyramid ABCD. The pyramid FFGH is therefore greater than the sum of the prisms described about it, which is impossible. The pyramids ABCD, EFGH, therefore, are not unequal, that is, they are equal to one another.

PROPOSITION XVII. THEOREM.

Every prism having a triangular base may be divided into three pyramids that have triangular bases, and that are equal to one another.

Let there be a prism of which the base is the triangle ABC, and let DEF be the triangle opposite to the base. The prism ABCDEF may be divided into three equal pyramids having triangular bases.

Join AE, EC, CD; and because ABED is a parallelogram, of which AE is the diameter, the triangle ADE is equal (Pl. Ge. I. 34) to the triangle ABE; therefore the pyramid of which the base is the triangle ADE, and vertex the point C, is equal (II. 16) to the pyramid, of which the base is the triangle ABE, and vertex the point C. But the pyramid of which the base is the triangle ABE, and vertex the point C, that is, the pyramid ABCE is equal to the pyramid DEFC, for they have equal bases, namely, the triangles ABC, DFE, and the same altitude, namely, the altitude of



the prism ABCDEF. Therefore the three pyramids ADEC, ABEC, DFEC, are equal to one another. But the pyramids ADEC, ABEC, DFEC, make up the whole prism ABCDEF; therefore the prism ABCDEF is divided into three equal pyramids.

COR. 1.—From this it is manifest that every pyramid is the third part of a prism which has the same base, and the same altitude with it; for if the base of the prism be any other figure than a triangle, it may be divided into prisms having triangular bases.

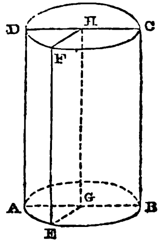
COR. 2.—Pyramids of equal altitudes are to one another as their bases; because the prisms upon the same bases, and of the same altitude, are (II. 10, Cor. 1) to one another as their bases.

PROPOSITION XVIII. THEOREM.

If from any point in the circumference of the base of a cylinder a straight line be drawn perpendicular to the plane of the base, it will be wholly in the cylindric superficies.

Let ABCD be a cylinder, of which the base is the circle AEB, DFC the circle opposite to the base, and GH the axis; from E, any point in the circumference AEB, let EF be drawn perpendicular to the plane of the circle AEB; the straight line EF is in the superficies of the cylinder.

Let F be the point in which EF meets the plane DFC opposite to the base; join EG and FH; and let AGHD be the rectangle (II. Def. 13), by the revolution of which the cylinder ABCD is described.



Now, because GH is at right angles to GA, the straight line which by its revolution describes the circle AEB, it is at right angles to all the straight lines in the plane of that circle which meet it in G, and it is therefore at right angles to the plane of the circle AEB. But EF is at right angles to the same plane; therefore EF and GH are parallel (I. 6), and in the same plane. And since the plane through GH and EF cuts the parallel planes AEB, DEC, in the straight lines EG and FH, EG is parallel to FH (I. 14). The

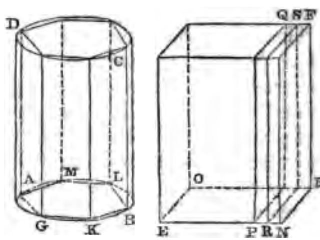
figure EGHF is therefore a parallelogram; and it has the angle EGH a right angle; therefore it is a rectangle, and is equal to the rectangle AH, because EG is equal to AG. Therefore, when in the revolution of the rectangle AH, the straight line AG coincides with EG, the two rectangles AH and EH will coincide, and the straight line AD will coincide with the straight line EF. But AD is always in the superficies of the cylinder, for it describes that superficies; therefore EF is also in the superficies of the cylinder.

PROPOSITION XIX. THEOREM.

A cylinder and a parallelopiped having equal bases and altitudes are equal to one another.

Let ABCD be a cylinder, and EF a parallelopiped having equal bases, namely, the circle AGB and the parallelogram EH, and having also equal altitudes; the cylinder ABCD is equal to the parallelopiped EF.

If not, let them be unequal; and first, let the cylinder be less than the parallelopiped EF; and from the parallelopiped EF let there be cut off a part EQ by a plane PQ parallel to NF, equal to the cylinder ABCD. In the circle AGB inscribe the polygon AGKBLM that shall differ from the circle by a space less than the parallelogram PH, and cut off from the parallelogram EH, a part OR equal to the polygon AGKBLM. The point R will fall between P and N. On the polygon AGKBLM let an upright prism AGBCD be constituted of the same altitude with the cylinder, which will therefore be less than the cylinder, because it is within it; and if through the point R a plane RS parallel to NF be made to pass, it will cut off the parallelopiped ES equal to the prism AGBC, because its base is equal to that of the prism, and its altitude is the same. But the prism AGBC is less than the cylinder ABCD, and the cylinder ABCD is equal to the parallelopiped EQ, by hypothesis; therefore, ES is



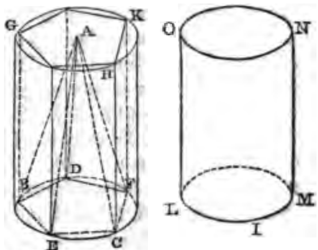
less than EQ, and it is also greater, which is impossible. The cylinder ABCD, therefore, is not less than the parallelopiped EF; and in the same manner it may be shown not to be greater than EF.

PROPOSITION XX. THEOREM.

If a cone and a cylinder have the same base and the same altitude, the cone is the third part of the cylinder.

Let the cone ABCD, and the cylinder BFKG, have the same base, namely, the circle BCD, and the same altitude, namely, the perpendicular from the point A upon the plane BCD, the cone ABCD is the third part of the cylinder BFKG.

If not, let the cone ABCD be the third part of another cylinder LMNO, having the same altitude with the cylinder BFKG, but let the bases BCD and LIM be unequal; and first, let BCD be greater than LIM.



Then, because the circle BCD is greater than the circle LIM, a polygon may be inscribed in BCD, that shall differ from it less than LIM does (Pl. Ge. Quad. 6), and which, therefore, will be greater than LIM. Let this be the polygon BECFD; and upon BECFD let there be constituted the pyramid ABECFD, and the prism BCFKHG.

Because the polygon BECFD is greater than the circle LIM, the prism BCFKHG is greater than the cylinder LMNO, for they have the same altitude, but the prism has the greater base. But the pyramid ABECFD is the third part of the prism (II. 17) BCFKHG; therefore it is greater than the third part of the cylinder LMNO. Now, the cone ABECFD is, by hypothesis, the third part of the cylinder LMNO; therefore, the pyramid ABECFD is greater than the cone ABCD, and it is also less, because it is inscribed in the cone, which is impossible; therefore the cone ABCD is not less than the third part of the cylinder BFKG. And

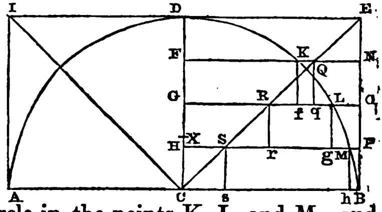
in the same manner, by circumscribing a polygon about the circle BCD, it may be shown that the cone ABCD is not greater than the third part of the cylinder BFKG; therefore it is equal to the third part of that cylinder.

PROPOSITION XXI. THEOREM.

If a hemisphere and a cone have equal bases and altitudes, a series of cylinders may be inscribed in the hemisphere, and another series may be described about the cone, having all the same altitudes with one another, and such that their sum shall differ from the sum of the hemisphere, and the cone by a solid less than any given solid.

Let ADB be a semicircle, of which the centre is C, and let CD be at right angles to AB; let DB and DA be squares described on DC, draw DE, and let the figure thus constructed revolve about DC; then the sector BCD, which is the half of the semicircle ADB, will describe a hemisphere having C for its centre (II. Def. 10), and the triangle CDE will describe a cone, having its vertex at C, and having for its base the circle (II. Def. 12) described by DE, equal to that described by BC, which is the base of the hemisphere. Let W be any given solid. A series of cylinders may be inscribed in the hemisphere ADB, and another described about the cone ECL, so that their sum shall differ from the sum of the hemisphere and the cone, by a solid less than the solid W.

Upon the base of the hemisphere let a cylinder be constituted equal to W, and let its altitude be CX. Divide CD into such a number of equal parts, that each of them shall be less than CX; let these be CH, HG, GF, and FD. Through the points F, G, H, draw FN, GO, HP, parallel



to CB, meeting the circle in the points K, L, and M; and the straight line CE in the points Q, R, and S. From the points K, L, M, draw Kf, Lg, Mh, perpendicular to GO,

HP, and CB; and from Q, R, and S, draw Qq, Rr, Ss, perpendicular to the same lines. It is evident that the figure being thus constructed, if the whole revolve about CD, the rectangles Ff, Gg, Hh, will describe cylinders (II. Def. 13) that will be circumscribed by the hemisphere BDA; and that the rectangles DN, Fq, Gr, Hs, will also describe cylinders that will circumscribe the cone ICE. Now, it may be demonstrated, as was done of the prisms inscribed in a pyramid (II. 15), that the sum of all the cylinders described within the hemisphere, is exceeded by the hemisphere by a solid less than the cylinder generated by the rectangle HB, that is, by a solid less than W, for the cylinder generated by HB is less than W. In the same manner, it may be demonstrated that the sum of the cylinders circumscribing the cone ICE is greater than the cone by a solid less than the cylinder generated by the rectangle DN, that is, by a solid less than W. Therefore, since the sum of the cylinders inscribed in the hemisphere, together with a solid less than W, is equal to the hemisphere; and since the sum of the cylinders described about the cone is equal to the cone together with a solid less than W; adding equals to equals, the sum of all these cylinders, together with a solid less than W, is equal to the sum of the hemisphere and the cone together with a solid less than W. Therefore, the difference between the whole of the cylinders and the sum of the hemisphere and the cone, is equal to the difference of two solids, which are each of them less than W; but this difference must also be less than W; therefore the difference between the two series of cylinders, and the sum of the hemisphere and cone, is less than the given solid W.

Or thus: let $I =$ the sum of the interior cylinders within the hemisphere; $E =$ the sum of the exterior without the cone; and Y and Z two solids each less than W. Then $I + Y = H$, and $E = C + Z$; therefore, adding equals to equals, $I + E + Y = H + C + Z$. Hence the difference between $(I + E)$ and $(H + C)$ is equal to that between Z and Y; but as these two solids are each less than W, their difference is still less than W; and hence also the difference between $(I + E)$ and $(H + C)$ is less than W.

PROPOSITION XXII. THEOREM.

The same things being supposed as in the last proposition, the sum of all the cylinders inscribed in the hemisphere, and described about the cone, is equal to a cylinder, having the same base and altitude with the hemisphere.

Let the figure DCB be constructed as before, and supposed to revolve about CD; the cylinders inscribed in the hemisphere, that is, the cylinders described by the revolution of the rectangles Hh, Gg, Ff, together with those described about the cone, that is, the cylinders described by the revolution of the rectangles Hs, Gr, Fq, and DN, are equal to the cylinder described by the revolution of the rectangle DB.

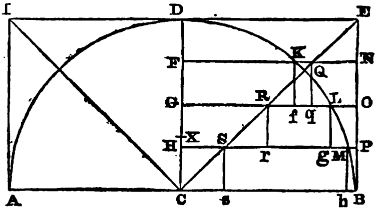
Let L be the point in which GO meets the circle ADB; then, because CGL is a right angle if CL be joined, the circles described with the distances CG and GL are equal to the circle described with the distance CL (Pl. Ge. Quad. 8, Cor. 2) or GO; now CG is equal to GR, because CD is equal to DE, and therefore also, the circles described with the distances GR and GL are together equal to the circle described with the distance GO, that is, the circles described by the revolution of GR and GL about the point G, are together equal to the circle described by the revolution of GO about the same point G; therefore also, the cylinders that stand upon the two first of these circles having the common altitude GH, are equal to the cylinder which stands on the remaining circle, and which has the same altitude GH. The cylinders described by the revolution of the rectangles Gg and Gr are therefore equal to the cylinder described by the rectangle GP. And as the same may be shown of all the rest, therefore the cylinders described by the rectangles Hh, Gg, Ff, and by the rectangles Hs, Gr, Fq, DN, are together equal to the cylinder described by DB, that is, to the cylinder having the same base and altitude with the hemisphere.

PROPOSITION XXIII. THEOREM.

Every sphere is two thirds of the circumscribing cylinder.

Let the figure be constructed as in the two last proposi-

tions, and if the hemisphere described by BDC be not equal to two thirds of the cylinder described by BD, let it be greater by the solid W. Then, as the cone described by CDE is one third of the cylinder (II. 20) described by BD, the cone and the hemisphere together will exceed the cylinder by W. But that cylinder is equal to the sum of all the cylinders described by the rectangles Hh, Gg, Ff, Hs, Gr, Fq, DN (II. 22); therefore the hemisphere and the cone added together exceed the sum of all these cylinders by the given solid W; which is absurd, for it has been shown that the hemisphere and the cone together differ from the sum of the cylinders by a solid less than W. The hemisphere is therefore equal to two thirds of the cylinder described by the rectangle BD; and therefore the whole sphere is equal to two thirds of the cylinder described by twice the rectangle BD, that is, to two thirds of the circumscribing cylinder.



SPHERICAL GEOMETRY.

DEFINITIONS.

1. A *sphere* is a solid conceived to be generated by the revolution of a semicircle about its diameter.

2. The centre of the semicircle is equally distant from every point on the surface of the sphere, and is therefore called the *centre* of the sphere.

3. Circles of the sphere, whose planes pass through the centre, are called *great* circles, and all others *small* circles.

4. A straight line, drawn through the centre of any circle of the sphere, perpendicular to its plane, and limited on both sides, by the surface of the sphere, is called the axis of that circle.

5. The *poles* of a circle of the sphere are the extremities of its axis.

6. By the *distance* of two points on the surface of the sphere is meant an arc of a great circle intercepted between them.

7. A *spherical angle* is that formed on the surface of the sphere by arcs of two great circles meeting at the angular point, and is measured by the inclination of the planes of the circles.

8. A *spherical triangle* is a figure formed on the surface of the sphere by arcs of three great circles, called its *sides*, each of which is less than a semicircle.

9. A quadrantal *triangle* is that of which one of the sides is a quadrant.

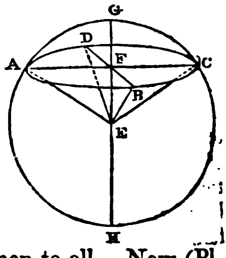
10. A *lunary surface* is a part of the surface of the sphere, contained by the halves of two great circles.

11. A *segment* of a sphere is a part cut off by a plane.

PROPOSITION I.

Every section of a sphere is a circle.

Let a plane cut the sphere AGH in any direction. If it pass through the centre, the section is evidently a circle. But if it do not pass through the centre, let ABCD be the section, and from E, the centre of the sphere, let EF be drawn perpendicular to its plane; also, let FA, FB, FC, be drawn in the plane, to meet the surface of the sphere. Then EA, EB, EC, being joined, the right-angled triangles EAF, EBF, ECF, have equal hypothenuses, EA, EB, EC, because they are radii of the sphere, and one side EF common to all. Now (Pl. Ge. I. 47) $EF^2 + FB^2 = EB^2 = EC^2 = EF^2 + FC^2$, and taking away the common part EF^2, there remains $FB^2 = FC^2$, or $FB = FC$. It is similarly proved that $FB = FA$. Consequently ABCD is a circle, whose centre is F.



COR. 1.—*Any two great circles cut one another in a diameter of the sphere, and therefore mutually bisect each other.*

COR. 2.—Only one great circle can pass through the same two points, in the surface of the sphere, that are not diametrically opposite.

For the plane of the great circle must pass through these two points and through the centre of the sphere (Sp. Ge. Def. 3), and only one plane can do so (So. Ge. I. 2, Cor. 2).

COR. 3.—Any two sides of a spherical triangle being produced, intersect again at the distance of a semicircle.

COR. 4.—The two poles of any circle, its centre, and the centre of the sphere, are always in the same straight line, and that straight line is perpendicular to the plane of the circle.

COR. 5.—And, therefore, if a line or plane be perpendicular to a circle of the sphere, and pass through one of these points, it will pass through the other three. Or, if it pass through two of them, it will be perpendicular to the circle, and also pass through the remaining two.

COR. 6.—Hence, two great circles, whose planes are perpendicular, pass through each other's poles; and conversely.

COR. 7.—And, if one great circle pass through a pole of another, the latter will pass through the poles of the former.

COR. 8.—All parallel circles have the same axis and the same poles; and conversely.

Schol.—It appears by the fourth and fifth corollaries, that, of these five conditions—of passing through the two poles of a circle of the sphere, through the centre of the circle, through the centre of the sphere, and of being perpendicular to the plane of the circle—if a straight line or a plane fulfil any two, it will also satisfy the other three.

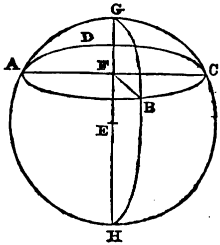
PROPOSITION II.

Each pole of any circle of the sphere is equally distant on the surface from every point in its circumference.

Let ABCD be any circle of the sphere, whose centre is E, and axis GFH. Its poles G, H, are each of them equally distant from its circumference.

For, let the sphere be cut by planes passing through G, H,

and let the sections—which will be great circles, because the line GH passes through the centre of the sphere—meet ABCD in the lines of common section FA, FB, FC. Then GA, GB, GC, being joined, the right-angled triangles GFA, GFB, GFC, have the sides FA, FB, FC, equal, because they are radii of the same circle, and one side GF common to all; and the angles at F are right angles (Sp. Ge. I. Cor. 4); therefore (Pl. Ge. I. 4) the hypothenuses GA, GB, GC, and consequently the arcs which they subtend, are likewise equal.



Cor. 1.—The pole of a great circle is at the distance of a quadrant from its circumference.

Cor. 2.—Hence any plane passing through the centre of the sphere, divides it into two equal parts, which are therefore called hemispheres.

Cor. 3.—If a point in the surface of the sphere be at the distance of a quadrant from other two points not diametrically opposite, it will be the pole of the great circle passing through them.

For only one great circle can pass through these two points, and its pole is distant from them by a quadrant. (Sp. Ge. I. Cor. 2, and II. Cor. 1.)

Cor. 4.—The radius of a small circle is the sine of its distance from either pole to the radius of the sphere, or the cosine of its distance from the parallel great circle.

Cor. 5.—Hence, those small circles, whose planes are equally distant from the centre, are equal; and conversely; and of two circles unequally distant, that which is nearer the centre is the greater; and conversely.

Cor. 6.—Parallel circles intercept equal arcs on those great circles which pass through their poles.

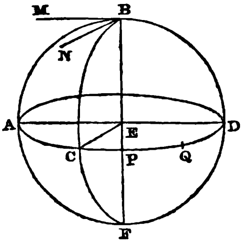
PROPOSITION III.

The intercepted arc of a great circle, whose pole is the *angular point,* is the measure of a spherical angle.

Let ABC be a spherical angle, of which the angular point

B is the pole of the great circle ACD. Then is the intercepted arc AC the measure of ABC.

For, let the tangents MB, NB, and the radii of the sphere EA, EB, EC, be drawn. The angle MBN is the same with the spherical angle ABC, for the tangents are perpendicular to BE (So. Ge. I. Def. 4, and Sp. Ge. Def. 7); but MBN is equal to AEC; because, since AB, BC, are quadrants, and BEA, BEC, right angles, MB, BN, are parallel to AE, EC. Wherefore the spherical angle ABC is equal to AEC, the measure of which is the arc AC to the radius of the sphere.



Cor. 1.—The circumferences of two great circles cut each other at right angles, when their planes are perpendicular; and conversely.

Cor. 2.—At the point of intersection of two great circles, the opposite angles are equal, the two adjacent angles are together equal to two right angles, and each angle is equal to its opposite one, at the other point of intersection (Sp. Ge. Def. 7).

Cor. 3.—The distance of the adjacent poles of two great circles is the measure of their inclination, or of the spherical angle.

For, since AB, BC, pass through the poles of ACD, ACD passes through the poles of AB, BC (Sp. Ge. I. Cor. 7); let P be the pole of AB, and Q the adjacent pole of BC; then AP, CQ, are quadrants, and AC = PQ.

Cor. 4.—The intercepted arc of any circle, whose pole is the angular point, is the measure of the spherical angle to the radius of that circle.

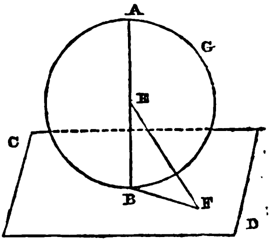
Cor. 5.—Two great circles, which pass through the poles of parallel circles, intercept similar arcs.

Cor. 6.—A spherical angle is equal to the inclination of two tangents, to the arcs containing it, drawn from the angular point.

PROPOSITION IV.

If a plane be perpendicular to the diameter of a sphere, at one of its extremities, it touches the sphere.

Let the plane CD be perpendicular to AB, the diameter of the sphere ABG, at its extremity B; then CD touches the sphere in that point.



For, let F be any other point in CD, and EF, FB, be drawn; the angle EBF is, by hypothesis, a right angle. Hence EF is greater than EB, and consequently F a point without the sphere. Thus the plane CD meets the sphere only in the point B, and therefore touches it.

COR. 1.—A sphere and a plane can touch one another only in one point.

COR. 2.—If a plane touch a sphere, the radius at the point of contact is perpendicular to it.

COR. 3.—If a plane touch a sphere, a perpendicular to it, at the point of contact, passes through the centre.

COR. 4.—If a plane touch a sphere, its line of common section, with the plane of any circle of the sphere passing through the point of contact, is a tangent to that circle.

For this line of common section is in the plane of the circle, and it touches the circle.

COR. 5.—A tangent, to any circle of the sphere, is the common tangent of all the circles in whose plane it is.

PROPOSITION V.

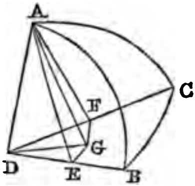
In isosceles spherical triangles, the angles at the base are equal.

Let ABC be a spherical triangle, having the side AB equal to the side AC; the spherical angles ABC and ACB are equal.

Let D be the centre of the sphere; join DB, DC, DA; and from A on the straight lines DB, DC, draw the perpen-

diculars AE, AF; and from the points E and F draw in the plane DBC the straight lines EG, FG, perpendicular to DB and DC, meeting one another in G; join AG.

Because DE is at right angles to each of the straight lines AE, EG, it is at right angles to the plane AEG, which passes through AE, EG (So. Ge. I. 4); and, therefore, every plane that passes through DE is at right angles to the plane AEG (So. Ge. I. 17); wherefore, the plane DBC is at right angles to the plane AEG. For the same reason, the plane DBC is at right angles to the plane AFG; and therefore AG, the common section of the planes AFG, AEG, is at right angles (So. Ge. I. 18) to the plane DBC, and the angles AGE, AGF, are consequently right angles.



But, since the arc AB is equal to the arc AC, the angle ADB is equal to the angle ADC. Therefore the triangles ADE, ADF, have the angles EDA, FDA, equal, as also the angles AED, AFD, which are right angles; and they have the side AD common; therefore the other sides are equal, viz. AE to AF (Pl. Ge. I. 26), and DE to DF. Again, because the angles AGE, AGF, are right angles, the squares on AG and GE are equal to the square of AE; and the squares of AG and GF to the square of AF. But the squares of AE and AF are equal; therefore the squares of AG and GE are equal to the squares of AG and GF; and taking away the common square of AG, the remaining squares of GE and GF are equal, and GE is therefore equal to GF. Wherefore, in the triangles AFG, AEG, the side GF is equal to the side GE, and AF has been proved to be equal to AE, and the base AG is common; therefore the angle AFG is equal to the angle AEG (Pl. Ge. I. 8). But the angle AFG is the angle which the plane ADC makes with the plane DBC (So. Ge. I. Def. 4), because FA and FG, which are drawn in these planes, are at right angles to DF, the common section of the planes. The angle AFG (Sp. Ge. Def. 7) is therefore equal to the spherical angle ACB; and, for the same reason, the angle AEG is equal to the *spherical angle ABC*. But the angles AFG, AEG, are

equal. Therefore the spherical angles ACB, ABC, are also equal.

PROPOSITION VI.

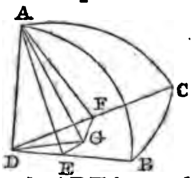
If the angles at the base of a spherical triangle be equal, the triangle is isosceles.

Let ABC be a spherical triangle having the angles ABC, ACB, equal to one another; the sides AC and AB are also equal.

Let D be the centre of the sphere; join DB, DC, DA, and from A on the straight lines DB, DC, draw the perpendiculars AE, AF; and from the points E and F, draw in the plane DBC the straight lines EG, FG, perpendicular to DB and DC, meeting one another in G; join AG.

Then, it may be proved, as was done in the last proposition, that AG is at right angles to the plane BCD, and that therefore the angles AGF, AGE, are right angles, and also that the angles AFG, AEG, are equal to the angles which the planes DAC, DAB, make with the plane DBC. But because the spherical angles ACB, ABC, are equal, the angles which the planes DAC, DAB, make with the plane DBC, are equal (Sp. Ge. Def. 7), and therefore the angles AFG, AEG, are also equal. The triangles AGE, AGF, have therefore two angles of the one equal to two angles of the other, and they have also the side AG common; wherefore they are equal, and the side AF is equal to the side AE.

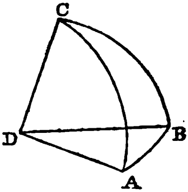
Again, because the triangles ADF, ADE, are right angled at F and E, the squares of DF and FA are equal to the square of DA, that is, to the squares of DE and EA; now, the square of AF is equal to the square of AE, therefore the square of DF is equal to the square of DE, and the side DF to the side DE. Therefore in the triangles DAF, DAE, because DF is equal to DE, and DA common, and also AF equal to AE, the angle ADF is equal to the angle ADE; therefore, also, the arcs AC and AB, *which are the* measures of the angles ADF and ADE, are *equal to one another*; and the triangle ABC is isosceles.



PROPOSITION VII.

Any two sides of a spherical triangle are greater than the third.

Let ABC be a spherical triangle, any two sides AB, BC, are greater than the third side AC.



Let D be the centre of the sphere; join DA, DB, DC.

The solid angle at D is contained by three plane angles ADB, ADC, BDC (So. Ge. II. 1); any two of which ADB, BDC, are greater than the third ADC; that is, any two sides AB, BC, of the spherical triangle ABC, are greater than the third AC.

PROPOSITION VIII.

The three sides of a spherical triangle are less than a circle.

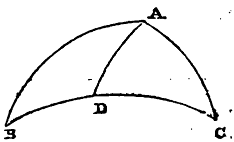
Let ABC be a spherical triangle as before, the three sides AB, BC, AC, are less than a circle.

Let D be the centre of the sphere. The solid angle at D is contained by three plane angles BDA, BDC, ADC, which together are less than four right angles (So. Ge. II. 2); therefore the sides AB, BC, AC, together, will be less than four quadrants, that is, less than a circle.

PROPOSITION IX.

In a spherical triangle the greater angle is opposite to the greater side; and conversely.

Let ABC be a spherical triangle, the greater angle A is opposed to the greater side BC.



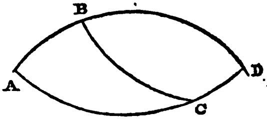
Let the angle BAD be made equal to the angle B, and then BD, DA, will be equal (Sp. Ge. 6), and therefore AD, DC, are equal to BC; but AD, DC, are greater than AC (Sp. Ge. 7); therefore BC is greater than AC, that is, the greater angle A is opposite to the greater side BC. The converse is demonstrated as in Pl. Ge. I. 19.

PROPOSITION X.

In a spherical triangle, according as the sum of two of the sides is greater than a semicircle, equal to it, or less, the interior angle at the base is greater than the exterior and opposite angle at the base, equal to it, or less; and the sum of the two interior angles at the base greater than two right angles, equal to two right angles, or less than two right angles.

Let ABC be a spherical triangle, of which the sides are AB and BC; produce the side AB and the base AC till they meet again in D; then, the arc ABD is a semicircle, and the spherical angles at A and D are equal, because each of them is the inclination of the circle ABD to the circle ACD.

1. If AB, BC, be equal to a semicircle, that is, to AD, BC will be equal to BD, and therefore (Sp. Ge. 5) the angle D, or the angle A, will be equal to the angle BCD.



2. If AB, BC, together be greater than a semicircle, that is, greater than ABD, BC will be greater than BD; and therefore (Sp. Ge. 9) the angle D, that is, the angle A, is greater than the angle BCD.

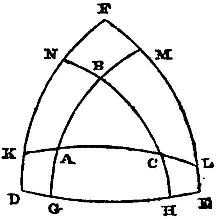
3. In the same manner, it is shown if AB, BC, together be less than a semicircle, that the angle A is less than the angle BCD. And since the angles BCD, BCA, are equal to two right angles, if the angle A be greater than BCD, A and ACB together will be greater than two right angles. If A be equal to BCD, A and ACB together will be equal to two right angles; and if A be less that BCD, A and ACB will be less than two right angles.

PROPOSITION XI.

If the angular points of a spherical triangle be made the poles of three great circles, these three circles by their intersections will form a triangle which is said to be supplemental to the former; and the two triangles are such, that the sides of the one are the supplements of the arcs which measure the angles of the other.

Let ABC be a spherical triangle; and from the points A, B, and C, as poles, let the great circles FE, ED, DF, be described, intersecting one another in F, D, and E; the sides of the triangle FED are the supplements of the measures of the angles A, B, C; namely, FE of the angle BAC, DE of the angle ABC, and DF of the angle ACB. And again, AC is the supplement of the angle DFE, AB of the angle FED, and BC of the angle EDF.

Let AB produced meet DE, EF, in G, M; let AC meet FD, FE, in K, L; and let BC meet FD, DE, in N, H.



Since A is the pole of FE, and the circle AC passes through A, EF will pass through the pole of AC (Sp. Ge. I. Cor. 7); and since AC passes through C, the pole of FD, FD will pass through the pole of AC; therefore the pole of AC is in the point F, in which the arcs DF, EF, intersect each other. In the same manner, D is the pole of BC, and E the pole of AB.

And since F, E, are the poles of AL, AM, the arcs FL and EM are quadrants, and FL, EM, together, that is, FE and ML together, are equal to a semicircle. But since A is the pole of ML, ML is the measure of the angle BAC (Sp. Ge. 3), consequently FE is the supplement of the measure of the angle BAC. In the same manner, ED, DF, are the supplements of the measures of the angles ABC, BCA.

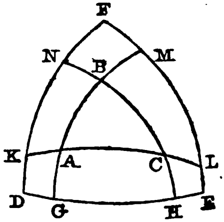
Since likewise CN, BH, are quadrants, CN, BH, together, that is, NH, BC, together, are equal to a semicircle; and since D is the pole of NH, NH is the measure of the angle FDE, therefore the measure of the angle FDE is the supplement of the side BC. In the same manner, it is shown that the measures of the angles DEF, EFD, are the supplements of the sides AB, AC, in the triangle ABC.

Schol.—The triangles ABC, DEF, are called *polar* triangles.

PROPOSITION XII.

The three angles of a spherical triangle are greater than two right angles, and less than six right angles.

The measures of the angles A, B, C, in the triangle ABC, together with the three sides of the supplemental triangle DEF, are (Sp. Ge. 11) equal to three semicircles; but the three sides of the triangle FDE are (Sp. Ge. 8) less than two semicircles; therefore the measures of the angles A, B, C, are greater than a semicircle; and hence the angles A, B, C, are greater than two right angles.



And because all the external and internal angles of any triangle are equal to six right angles; therefore, all the internal angles are less than six right angles.

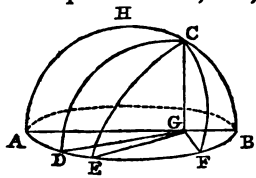
PROPOSITION XIII.

If to the circumference of a great circle, from a point which is not the pole of it, arcs of great circles be drawn; the greatest of these arcs is that which passes through the pole of the first-mentioned circle, and the supplement of it is the least; and of the others, that which is nearer to the greatest is greater than that which is more remote.

Let ADB be the circumference of a great circle, of which the pole is H, and let C be any other point; through C and H let the semicircle ACB be drawn meeting the circle ADB in A and B; and let the arcs CD, CE, CF, also be described. From C draw CG perpendicular to AB, and then, because the circle AHCB which passes through H, the pole of the circle ADB, is at right angles to ADB, CG is perpendicular to the plane ADB. Join GD, GE, GF, CD, CE, CF, CA, CB.

Of all the straight lines drawn from G to the circumference ADB, GA is the greatest, and GB the least (Pl. Ge. III. 7); and GD, which is nearer to GA, is greater than GE, which is more remote. But the triangles CGA, CGD, are right angled at G, and they have the common side CG; therefore the squares of CG, GA, together, that

is, the square of CA is greater than the squares of CG, GD, together, that is, than the square of CD; therefore CA is greater than CD, and the arc CA than the arc CD. In the same manner, since GD is greater than GE, and GE than GF, it is shown that CD is greater than CE, and CE than CF, and, consequently, the arc CD greater than the arc CE, and the arc CE greater than the arc CF. Also, because AG is the greatest, and GB the least, of all the lines drawn from G, CA is the greatest, and CB the least, of all the lines drawn from C, and therefore the arc CA is the greatest, and CB, its supplement, the least of all the arcs drawn through C.

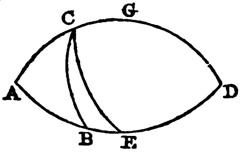


PROPOSITION XIV.

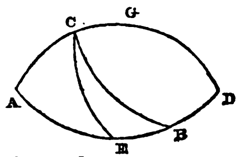
In a right-angled spherical triangle the sides are of the same affection with the opposite angles; that is, if the sides be greater or less than quadrants, the opposite angles will be greater or less than right angles.

Let ABC be a spherical triangle right angled at A, any side AB will be of the same affection with the opposite angle ACB.

Case 1. Let AB be less than a quadrant. Let AE be a quadrant, and EC an arc of a great circle passing through E, C. Since A is a right angle, and AE a quadrant, E is the pole of the great circle AC, and ECA a right angle; but ECA is greater than BCA, therefore BCA is less than a right angle.



Case 2. Let AB be greater than a quadrant; make AE equal to a quadrant, and let a great circle pass through C, E. ECA is a right angle as before, and BCA is greater than ECA, that is, greater than a right angle.



PROPOSITION XV.

If the two sides of a right-angled spherical triangle be of the same affection, the hypotenuse will be less than a quadrant; and if they be of different affection, the hypotenuse will be greater than a quadrant.

Let ABC (last figure) be a right-angled spherical triangle; if the two sides AB, AC, be of the same or of different affection, the hypotenuse BC will be less or greater than a quadrant.

Case 1. Let AB, AC, be each less than a quadrant. Let AE, AG, be quadrants; G will be the pole of AB, and E the pole of AC, and EC a quadrant; but (Sp. Ge. 13) CE is greater than CB, since CB is farther off from CGD than CE. In the same manner, it is shown that CB, in the triangle CBD, where the two sides CD, BD, are each greater than a quadrant, is less than CE, that is, less than a quadrant.

Case 2. Let AC be less, and AB greater than a quadrant; then the hypotenuse BC will be greater than a quadrant; for, let AE be a quadrant, then E is the pole of AC, and EC will be a quadrant. But CB is greater than CE (Sp. Ge. 13), since AC passes through the pole of ABD.

COR. 1.—Hence, conversely, if the hypotenuse of a right-angled triangle be greater or less than a quadrant, the sides will be of different or the same affection.

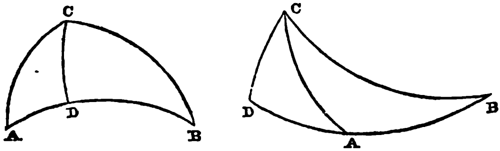
COR. 2.—Since (Sp. Ge. 14) the angles of a right-angled spherical triangle have the same affection with the opposite sides, therefore, according as the hypotenuse is greater or less than a quadrant, the angles will be of different or of the same affection.

PROPOSITION XVI.

In any spherical triangle, if the perpendicular upon the base from the opposite angle fall within the triangle, the angles at the base are of the same affection; and if the perpendicular fall without the triangle, the angles at the base are of different affection.

Let ABC be a spherical triangle, and let the arc CD be *drawn from C* perpendicular to the base AB.

1. Let CD fall within the triangle; then since ADC, BDC, are right-angled spherical triangles, the angles A, B, must each be of the same affection with CD (Sp. Ge. 14).



2. Let CD fall without the triangle; then (Sp. Ge. 14) the angle B is of the same affection with CD; and the angle CAD is of the same affection with CD; therefore the angles CAD and B are of the same affection, and the angles CAB and B of different affections.

COR.—Hence, if the angles A and B be of the same affection, the perpendicular will fall within the base; for, if it did not, A and B would be of different affection. And if the angles A and B be of opposite affection, the perpendicular will fall without the triangle; for, if it did not, the angles A and B would be of the same affection, contrary to the supposition.

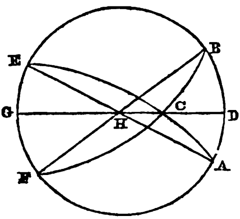
PROPOSITION XVII.

If to the base of a spherical triangle a perpendicular be drawn from the opposite angle, which either falls within the triangle, or is the nearest of the two that fall without; the least of the segments of the base is adjacent to the least of the sides of the triangle, or to the greatest, according as the sum of the sides is less or greater than a semicircle.

Let ABEF be a great circle of a sphere, H its pole, and GHD any circle passing through H, which therefore is perpendicular to the circle ABEF. Let A and B be two points in the circle ABEF on opposite sides of the point D, and let D be nearer to A than to B, and let C be any point in the circle GHD, between H and D. Through the points A and C, B and C, let the arcs AC and BC be drawn, and let them be produced till they meet the circle ABEF in the points E and F; then the arcs ACE, BCF, are semicircles. Also ACB, ACF, CFE, ECB, are four spherical triangles

contained by arcs of the same circles, and having the same perpendiculars CD and CG.

1. Now, because CE is nearer to the arc CHG than CB is, CE is greater than CB, and therefore CE and CA are greater than CB and CA; wherefore CB and CA are less than a semicircle; but because AD is, by supposition, less than DB, AC is also less than CB (Sp. Ge. 13); and therefore in this case, namely, when the perpendicular falls within the triangle, and when the sum of the sides is less than a semicircle, the least segment is adjacent to the least side.



2. Again, in the triangle FCA the two sides FC and CA are less than a semicircle; for, since AC is less than CB, AC and CF are less than BC and CF. Also, AC is less than CF, because it is more remote from CHG than CF is; therefore the least segment of the base AD is in this case also adjacent to the least side.

3. But in the triangle FCE the two sides FC and CE are greater than a semicircle; for, since FC is greater than CA, FC and CE are greater than AC and CE. And because AC is less than CB, EC is greater than CF, and EC is therefore nearer to the perpendicular CHG than CF is, wherefore EG is the least segment of the base, and is adjacent to the greater side.

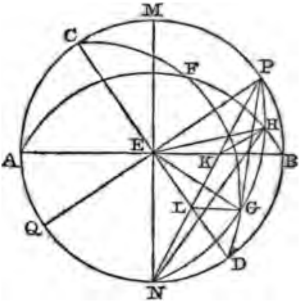
4. In the triangle ECB the two sides EC, CB, are greater than a semicircle; for, since by supposition CB is greater than CA, EC and CB are greater than EC and CA. Also, EC is greater than CB; wherefore in this case, also, the least segment of the base EG is adjacent to the greatest side of the triangle.

PROPOSITION XVIII.

Any two circles of the sphere, passing through the poles of two great circles, intercept equal arcs upon them.

Let AFB, CFD, be the two great circles intersecting one another in F, and let F be the pole of the great circle ACBD, cutting them in the diameters AEB, CED. The circle

ACBD passes through their poles; let the diameter MN be perpendicular to AB, and PQ to CD; then M, N, are the poles of AFB, and P, Q, the poles of CFD. Let the small circle PGN pass through the poles P, N, and cut the circle ACBD in the line of common section PKLN; the arcs BH, DG, of the circles AFB, CFD, intercepted by the circle passing through P, N, are equal.



For, let EH, HK, EG, GL, PG, NH, be drawn.

In the triangles PEL, NEK, angle P = N, for PE = NE; also NEK, PEL, being right angles, are equal; hence (Pl. Ge. I. 26) PL = NK, and EL = EK. Again, the quadrants PG, NH, are equal, and taking away HG, the arc PH = NG, and the chord PG = NH. Hence, in the triangles PLG, NKH, PL = NK, PG = NH, and angles LPG, KNH, standing on equal arcs NG, PH, are equal; hence, LG = KH. Again, in the triangles EKH, ELG, having their sides respectively equal, angle HEK = GEL, and hence the arc HB = GD.

SPHERICAL TRIGONOMETRY.

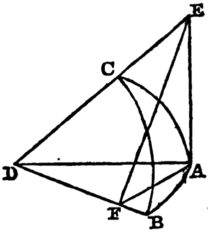
Spherical Trigonometry treats of those relations between the sides and angles of spherical triangles, by which their numerical values may be computed.

The trigonometrical lines defined in Plane Trigonometry are employed in reference to the sides and angles of spherical triangles.

PROPOSITION I.

In right-angled spherical triangles, the sine of either of the sides about the right angle, is to the radius of the sphere, as the tangent of the remaining side is to the tangent of the angle opposite to that side.

Let ABC be a triangle, having the right angle at A; and let AB be either of the sides, the sine of the side AB will be to the radius, as the tangent of the other side AC to the tangent of the angle ABC, opposite to AC. Let D be the centre of the sphere; join AD, BD, CD, and let AF be drawn perpendicular to BD, which therefore will be the sine of the arc AB, and from the point F, let there be drawn in the plane BDC the straight line FE at right angles to BD, meeting DC in E, and let AE be joined. Since therefore the straight line DF is at right angles to both FA and FE, it will also be at right angles to the plane AEF (So. Ge. I. 4); wherefore the plane ABD, which passes through DF, is perpendicular to the plane AEF (So. Ge. I. 17), and the plane AEF perpendicular to ABD; but the plane ACD or AED is also perpendicular to the same ABD; therefore the common section, namely, the straight line AE, is at right angles to the plane ABD (So. Ge. I. 18), and EAF, EAD, are right angles. Therefore, AE is the tangent of the arc AC; and in the rectilineal triangle AEF, having a right angle at A, AF is to the radius as AE to the tangent of the angle AFE (Pl. Tr. 2); but AF is the sine of the arc AB, and AE the tangent of the arc AC, and the angle AFE is the inclination of the planes CBD, ABD (So. Ge. I. Def. 7), or the spherical angle ABC; therefore the sine of the arc AB is to the radius as the tangent of the arc AC to the tangent of the opposite angle ABC.



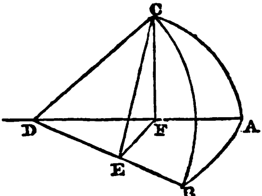
COR.—And since by this proposition the sine of the side AB is to the radius, as the tangent of the other side AC to the tangent of the angle ABC opposite to that side; and as the radius is to the cotangent of the angle ABC, so is the tangent of the same angle ABC to the radius (Pl. Tr. 4, Cor. to Def.); by equality, the sine of the side AB is to the cotangent of the angle ABC adjacent to it, as the tangent of the other side AC to the radius.

PROPOSITION II.

In right-angled spherical triangles the sine of the hypotenuse is to the radius, as the sine of either side is to the sine of the angle opposite to that side.

Let the triangle ABC be right angled at A, and let AC be either of the sides; the sine of the hypotenuse BC will be to the radius as the sine of the arc AC is to the sine of the angle ABC.

Let D be the centre of the sphere, and let CE be drawn perpendicular to DB, which will therefore be the sine of the hypotenuse BC; and from the point E let there be drawn in the plane ABD the straight line EF perpendicular to DB, and let CF be joined; CF will be at right angles to the plane ABD, as was shown in the preceding proposition of the straight line EA; wherefore CFD, CFE, are right angles, and CF is the sine of the arc AC; and in the triangle CFE, having the right angle CFE, CE is to the radius, as CF to the sine of the angle CEF (Pl. Tr. 1). But, since CE, FE, are at right angles to DEB, which is the common section of the planes CBD, ABD, the angle CEF is equal to the inclination of these planes (So. Ge. I. Def. 4), that is, to the spherical angle ABC. The sine, therefore, of the hypotenuse CB is to the radius as the sine of the side AC is to the sine of the opposite angle ABC.



COR.—Of these three, namely, the hypotenuse, a side, and the angle opposite to that side, any two being given, the third may be found.

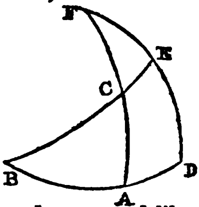
PROPOSITION III.

In right-angled spherical triangles, the cosine of the hypotenuse is to the radius as the cotangent of either of the angles is to the tangent of the remaining angle.

Let ABC be a spherical triangle, having a right angle at A, the cosine of the hypotenuse BC will be to the radius

as the cotangent of the angle ABC to the tangent of the angle ACB.

Describe the circle DE, of which B is the pole, and let it meet AC in F, and the circle BC in E; and since the circle BD passes through the pole B of the circle DF, DF will pass through the pole of BD (Sp. Ge. I. Cor. 7). And since AC is perpendicular to BD, AC will also pass through the pole of BD; wherefore, the pole of the circle BD is in the point where the circles AC, DE, intersect, that is, in the point F. The arcs FA, FD, are therefore quadrants, and likewise the arcs BD, BE. In the triangle CEF, right angled at the point E, CE is the complement of BC, the hypotenuse of the triangle ABC; EF is the complement of the arc ED, which is the measure of the angle ABC; FC, the hypotenuse of the triangle CEF, is the complement of AC; and the arc AD, which is the measure of the angle CFE, is the complement of AB.



But (Sp. Tr. 1) in the triangle CEF, the sine of the side CE is to the radius, as the tangent of the other side EF is to the tangent of the angle ECF opposite to it; that is, in the triangle ABC, the cosine of the hypotenuse BC is to the radius as the cotangent of the angle ABC is to the tangent of the angle ACB.

Cor. 1.—Of these three, namely, the hypotenuse and the two angles, any two being given, the third will also be given.

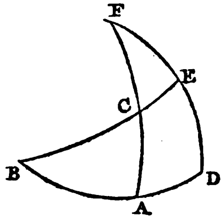
Cor. 2.—And since by this proposition the cosine of the hypotenuse BC is to the radius as the cotangent of the angle ABC to the tangent of the angle ACB, and since the radius is to the cotangent of ACB, as the tangent of ACB to the radius (Pl. Tr. 4, Cor. to Def.); therefore, by equality, the cosine of the hypotenuse BC is to the cotangent of the angle ACB, as the cotangent of the angle ABC to the radius.

Schol.—The triangle CEF is called the *complementary triangle.*

PROPOSITION IV.

In right-angled spherical triangles, the cosine of an angle is to the radius, as the tangent of the side adjacent to that angle is to the tangent of the hypotenuse.

The same construction remaining. In the triangle CEF (Sp. Tr. 1), the sine of the side EF is to the radius, as the tangent of the other side CE is to the tangent of the angle CFE opposite to it; that is, in the triangle ABC, the cosine of the angle ABC is to the radius as the cotangent of the hypotenuse BC to the cotangent of the side AB, adjacent to ABC, or as the tangent of the side AB to the tangent of the hypotenuse, since the tangents of two arcs are reciprocally proportional to their cotangents (Pl. Tr. 4, Cor. to Def.)



Cor.—And since by this proposition the cosine of the angle ABC is to the radius, as the tangent of the side AB is to the tangent of the hypotenuse BC; and as the radius is to the cotangent of BC, so is the tangent of BC to the radius; by equality, the cosine of the angle ABC will be to the cotangent of the hypotenuse BC, as the tangent of the side AB, adjacent to the angle ABC, to the radius.

PROPOSITION V.

In right-angled spherical triangles, the cosine of either of the sides is to the radius, as the cosine of the hypotenuse is to the cosine of the other side.

The same construction remaining. In the triangle CEF, the sine of the hypotenuse CF is to the radius, as the sine of the side CE to the sine of the opposite angle CFE (Sp. Tr. 2); that is, in the triangle ABC, the cosine of the side CA is to the radius as the cosine of the hypotenuse BC to the cosine of the other side BA.

PROPOSITION VI.

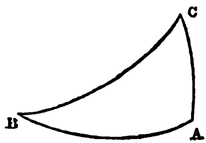
In right-angled spherical triangles, the cosine of either of the sides is to the radius, as the cosine of the angle opposite to that side is to the sine of the other angle.

The same construction remaining. In the triangle CEF, the sine of the hypotenuse CF is to the radius as the sine of the side EF is to the sine of the angle ECF opposite to it; that is, in the triangle ABC, the cosine of the side CA is to the radius, as the cosine of the angle ABC, opposite to it, is to the sine of the other angle ACB.

PROPOSITION VII.

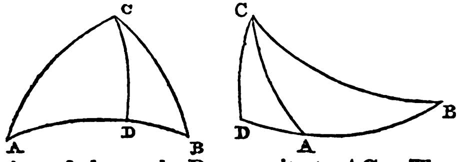
In spherical triangles, whether right-angled or oblique-angled, the sines of the sides are proportional to the sines of the angles opposite to them.

First, let ABC be a right-angled triangle, having a right angle at A; therefore (Sp. Tr. 2) the sine of the hypotenuse BC is to the radius (or the sine of the right angle at A) as the sine of the side AC to the sine of the angle B. And in like manner, the sine of BC is to the sine of the angle A, as the sine of AB to the sine of the angle C;



wherefore (Pl. Ge. V. 11) the sine of the side AC is to the sine of the angle B, as the sine of AB to the sine of the angle C.

Secondly, let ABC be an oblique-angled triangle, the sine of any of the sides BC, will be to the sine of any of the other two AC, as the sine of the angle A, opposite to BC, is



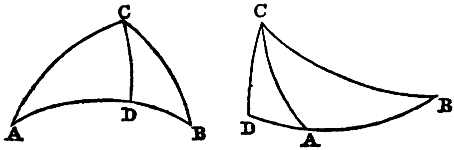
to the sine of the angle B, opposite to AC. Through the point C, let there be drawn an arc of a great circle CD perpendicular upon AB; and in the right-angled triangle BCD (Sp. Tr. 2), the sine of BC is to the radius, as the sine of CD to the sine of the angle B; and in the triangle ADC, by inversion, the radius is to the sine of AC as the sine of the angle A to the sine of DC; therefore, by indirect equality, the sine of BC is to the sine of AC, as the sine of the angle A to the sine of the angle B.

PROPOSITION VIII.

In oblique-angled spherical triangles, a perpendicular arc being drawn from any of the angles upon the opposite side, the cosines of the angles at the base are proportional to the sines of the segments of the vertical angle.

Let ABC be a triangle, and the arc CD perpendicular to the base BA; the cosine of the angle B will be to the cosine of the angle A, as the sine of the angle BCD to the sine of the angle ACD.

For (Sp. Tr. 6) the cosine of the angle B is to the sine of the angle BCD, as the cosine of the side CD is to the radius;



and also the cosine of the angle A to the sine of the angle ACD in the same ratio; therefore, by alternation, the cosine of the angle B is to the cosine of the angle A, as the sine of the angle BCD to the sine of the angle ACD.

PROPOSITION IX.

The same things remaining, the cosines of the sides BC, CA, are proportional to the cosines of BD, DA, the segments of the base.

For (Sp. Tr. 5) the cosine of BC is to the cosine of BD, as the cosine of DC to the radius, and the cosine of AC to the cosine of AD in the same ratio; wherefore, by alternation, the cosines of the sides BC, CA, are proportional to the cosines of the segments of the base BD, DA.

PROPOSITION X.

The same construction remaining, the sines of BD, DA, the segments of the base, are reciprocally proportional to the tangents of B and A, the angles at the base.

For (Sp. Tr. 1) the sine of BD is to the radius, as the tangent of DC to the tangent of the angle B; and also, the *radius to the sine of AD*, as the tangent of A to the tangent

of DC; therefore, by indirect equality, the sine of BD is to the sine of DA, as the tangent of A to the tangent of B.

PROPOSITION XI.

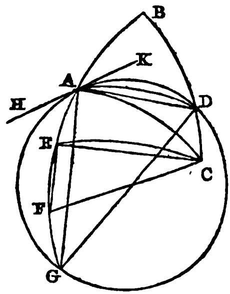
The same construction remaining, the cosines of the segments of the vertical angle are reciprocally proportional to the tangents of the sides.

For (Sp. Tr. 4) the cosine of the angle BCD is to the radius, as the tangent of CD is to the tangent of BC; and also (Sp. Tr. 4, by inversion), the radius is to the cosine of the angle ACD, as the tangent of AC to the tangent of CD; therefore, by indirect equality, the cosine of the angle BCD is to the cosine of the angle ACD, as the tangent of AC is to the tangent of BC.

PROPOSITION XII.

If, from the extremities of the base of any spherical triangle, arcs of great circles be described to meet the sides, and to cut off a part on each from the vertex, equal to the other side; the rectangle under the sines of half these arcs shall be equal to the rectangle under the sines of the excesses of the semiperimeter above the two sides.

Let ABC be a spherical triangle, having two unequal sides AB, BC. From BC, the greater, let BD be cut off equal to BA, and on BA produced make BE = BC; and let great circles pass through A, D, and E, C. Then if S denote half the sum of the sides, $\text{Sin}\,\frac{1}{2}\,\text{AD} \cdot \sin\frac{1}{2}\,\text{EC} = \text{Sin}\,(\text{S}-\text{AB}) \cdot \sin\,(\text{S}-\text{BC})$.



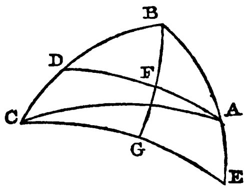
For, in BA produced, let AF = AC, and FG = AE; also, let the straight lines AG, GD, DA, EF, FC, CE, be drawn. The straight lines AD and EC are parallel; for, if a great circle bisect the angle at B, it also bisects the arcs AD, EC, and is perpendicular to their planes; therefore the cords AD, EC, are perpendicular to the lines of common section, and consequently both are perpendicular to the plane of that great circle, and are

parallel. But AG and EF are also parallel; therefore the plane of the triangle DAG is parallel to the plane of the triangle CEF. Let the plane AFC, which cuts the latter in FC, cut the former in HK. Then HK is parallel to FC (So. Ge. I. 14). Therefore, since the cords AF, AC, are equal, HK is a tangent to the circle that passes through A, F, C, and consequently it is also a tangent to the circle that passes through A, D, G (Sp. Ge. 4, Cor. 5). Hence the angle ADG = GAH = EFC. But the angle DAG = CEF (So. Ge. I. 9). Therefore the triangles AGD, EFC, are equiangular, and AD : AG = EF : EC, or $\frac{1}{2}$ AD : $\frac{1}{2}$ AG = $\frac{1}{2}$ EF : $\frac{1}{2}$ EC; or, considering the arcs, sin $\frac{1}{2}$ AD : sin $\frac{1}{2}$ AG = sin $\frac{1}{2}$ EF : sin $\frac{1}{2}$ EC; and hence Sin $\frac{1}{2}$ AD · sin $\frac{1}{2}$ EC = Sin $\frac{1}{2}$ AG · sin $\frac{1}{2}$ EF. But arc AG = AF + FG = AC + AE = AC + BC — AB, and $\frac{1}{2}$ AG = $\frac{1}{2}$ (AC + BC + AB — 2 AB) = S — AB. Also EF = AF — AE = AC — (BC — AB) = AC — BC + AB (Pl. Ge. Ad. II. 2), or $\frac{1}{2}$ EF = $\frac{1}{2}$ (AC + AB + BC — 2 BC) = S — BC. Hence, Sin $\frac{1}{2}$ AD · sin $\frac{1}{2}$ EC = Sin (S — AB) · sin (S — BC).

PROPOSITION XIII.

In any spherical triangle, the rectangle under the sines of the two sides is to the square of the radius, as the rectangle under the excesses of the semiperimeter above these sides, to the square of the sine of half the vertical angle.

Let ABC be the triangle, BD = BA, BE = BC, and BFG bisecting the vertical angle; and AD, EC, drawn as in the preceding figure; then BFG bisects AD, EC, and cuts them at right angles, for it passes through their pole. In the right-angled triangles ABF, EBG,



$$\text{Sin AB} : \text{R} = \sin \text{AF} : \sin \tfrac{1}{2}\text{B},$$

$$\text{Sin BE} : \text{R} = \sin \text{EG} : \sin \tfrac{1}{2}\text{B};$$

hence (Pl. Ge. VI. 23, Cor. 1) Sin AB · sin BC : R^2 = Sin AF · sin EG : $\sin^2 \frac{1}{2}$ B. But (Sp. Tr. 12) Sin AF · sin EG = Sin (S — AB) · sin (S — BC); therefore,

Sin AB · sin BC : R^2 = Sin (S — AB) · sin (S — BC) : $\sin^2 \frac{1}{2}$ B.

COR.—If the angles of the triangle be denoted by A, B, and C, and the sides opposite to them respectively by a, b, and c, and half the sum of the sides by s, then

$$\text{Sin}^2 \tfrac{1}{2} \text{B} = \frac{\sin (s - a) . \sin (s - c)}{\sin a . \sin c}, \text{ if R} = 1.$$

By changing B into C, and c into b, a similar formula is found for $\sin^2 \frac{1}{2}$ C; and also, in the same manner, for $\sin^2 \frac{1}{2}$ A.

Solution of the Cases of Right-Angled Spherical Triangles.

PROBLEM.

In a right-angled spherical triangle, of the three sides and three angles, any two being given, besides the right angle to find the other three.

This problem has sixteen cases, the solutions of which are contained in the following table, where ABC is any spherical triangle right-angled at A.

GIVEN.	SOUGHT.	SOLUTION.	
BC and B.	AC.	R : sin BC = sin B : sin AC, (2)	1
	AB.	R : cos B = tan BC : tan AB, (4)	2
	C.	R : cos BC = tan B : cot C, (3)	3
AC and C.	AB.	R : sin AC = tan C : tan AB, (1)	4
	BC.	cos C : R = tan AC : tan BC, (4)	5
	B.	R : cos AC = sin C : cos B, (6)	6
AC and B.	AB.	tan B : tan AC = R : sin AB, (1)	7
	BC.	sin B : sin AC = R : sin BC, (2)	8
	C.	cos AC : cos B = R : sin C, (6)	9
AC and BC.	AB.	cos AC : cos BC = R : cos AB, (5)	10
	B.	sin BC : sin AC = R : sin B, (2)	11
	C.	tan BC : tan AC = R : cos C, (4)	12
AB and AC.	BC.	R : cos AB = cos AC : cos BC, (5)	13
	B.	sin AB : R = tan AC : tan B, (1)	14
	C.	sin AC : R = tan AB : tan C, (1)	14
B and C.	AB.	sin B : cos C = R : cos AB, (6)	15
	AC.	sin C : cos B = R : cos AC, (6)	15
	BC.	tan B : cot C = R : cos BC, (3)	16

Table for determining when the things found in the preceding are less than a Quadrant (Sp. Ge. 14 and 15).

The angle or arc found is less than 90°.	
When B is less than 90°.	1
When BC and B are of the same affection.	2
When BC and B are of the same affection.	3
When C is less than 90°.	4
When AC and C are of the same affection.	5
When AC is less than 90°.	6
Ambiguous.	7
Ambiguous.	8
Ambiguous.	9
When AC and BC are of the same affection.	10
When AC is less than 90°.	11
When AC and BC are of the same affection.	12
When AB and AC are of the same affection.	13
When AC is less than 90°.	14
When AB is less than 90°.	14
When C is less than 90°.	15
When B is less than 90°.	15
When B and C are of the same affection.	16

Schol.—The rules for the cases of right-angled spherical trigonometry may be reduced to two, called Napier's Rules of the Circular Parts.

In a right-angled spherical triangle, the right angle is neglected, and the hypotenuse, the two angles, and the complements of the two sides, are called *circular parts*; and *any of these parts* being called the *middle part*; the two

adjacent to it, *adjacent* parts; and the two remaining parts, *opposite* parts; then, the rectangle under the radius and the cosine of the middle part, is equal to that under the cotangents of the adjacent parts, or the sines of the opposite parts. Or if the middle part be called M; the two adjacent parts, A and a; and the opposite parts O and o,

$$R \cdot \cos M = \cot A \cdot \cot a,$$
$$\text{or } R \cdot \cos M = \sin O \cdot \sin o.$$

Either of these rules may be converted into a proportion by Pl. Ge. VI. 16.

The cases marked ambiguous are those in which the thing sought has two values, and may either be equal to a certain angle, or to the supplement of that angle. Of these there are three, in all of which the things given are a side, and the angle opposite to it; and accordingly, it is easy to show that two right-angled spherical triangles may always be found, that have a side and the angle opposite to it the same in both, but of which the remaining sides, and the remaining angle of the one, are the supplements of the remaining sides and the remaining angle of the other, each of each.

Though the affection of the arc or angle found may in all the other cases be determined by the rules in the second of the preceding tables, it may be useful to remark, that all these rules, except two, may be reduced to one, namely, that when the thing found by the rules in the first table is either a tangent or a cosine; and when, of the tangents or cosines employed in the computation of it, one only belongs to an obtuse angle, the angle required is also obtuse.

Thus, in the 15th case, when cos AB is found, if C be an obtuse angle, because of cos C, AB must be obtuse; and in case 16, if either B or C be obtuse, BC is greater than 90°, but if B and C are either both acute, or both obtuse, BC is less than 90°.

It is evident that this rule does not apply when that which is found is the sine of an arc; and this, besides the three ambiguous cases, happens also in other two, namely, the 1st and 11th.

Solution of the Cases of Oblique-Angled Spherical Triangles.

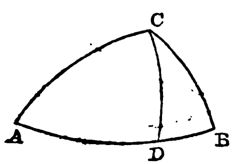
PROBLEM.

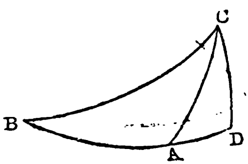
n any oblique-angled spherical triangle, of the three sides three angles, any three being given, the other three may ound.

n this table, the references (c. 4), (c. 5), &c. are to the s in the preceding tables.

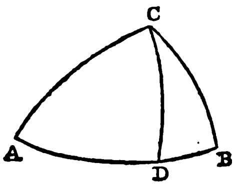
GIVEN.	SOUGHT.	SOLUTION.
'wo sides ꓥB, AC, ıd the in- ıded angle A.	1. One of the other angles, B.	Let fall the perpendicular CD from the unknown angle not required, on AB. R : cos A = tan AC : tan AD (c. 2); therefore BD is known, and sin BD : sin AD : : tan A : tan B (10); B and A are of the same or different affection, according as AB is greater or less than AD (Sp. Ge. 16).
	2. The third side BC.	Let fall the perpendicular CD from one of the unknown angles on the side AB. R : cos A = tan AC : tan AD (c. 2); therefore AD is known, and cos AD : cos BD : : cos AC : cos BC (9); according as the segments AD and DB are of the same or different affection, AC and CB will be of the same or different affection.

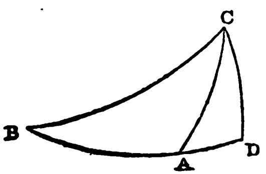
GIVEN.	SOUGHT.	SOLUTION.
Two angles A and ACB, and AC, the side between them.	3. The side BC.	From C the extremity of AC next the side sought, let fall the perpendicular CD on AB. R : cos AC : : tan A : cot ACD (c. 3); therefore BCD is known, and cos BCD : cos ACD : : tan AC : tan BC (11). BC is less or greater than 90°, according as the angles A and BCD are of the same or different affection.
	4. The third angle B.	Let fall the perpendicular CD from one of the given angles on the opposite side AB. R : cos AC : : tan A : cot ACD (c. 3); therefore the angle BCD is given, and sin ACD : sin BCD : : cos A : cos B (8); B and A are of the same or different affection, according as CD falls within or without the triangle, that is, according as ACB is greater or less than ACD (Sp. Ge. 16).





GIVEN.	SOUGHT.	SOLUTION.
	5. The angle B opposite to the other given side AC.	Sin BC : sin AC : : sin A : sin B (7). The affection of B is ambiguous, unless it can be determined by this rule, that according as AC + BC is greater or less than 180°, A + B is greater or less than 180° (Sp. Ge. 10).
wo sides ; and BC, l an angle A posite to e of them, BC.	6. The angle ACB contained by the given sides AC and BC.	From ACB, the angle sought, draw CD perpendicular to AB; then R : cos AC : : tan A : cot ACD (c. 3); and tan BC : tan AC : : cos ACD : cos BCD (11). ACD ± BCD = ACB, and ACB is ambiguous, because of the ambiguous sign + or —.
	7. The third side AB.	Let fall the perpendicular CD from the angle C contained by the given sides upon the side AB. R : cos A : : tan AC : tan AD (c. 5); cos AC : cos BC : : cos AD : cos BD (9). AB = AD ± BD; wherefore AB is ambiguous.





GIVEN.	SOUGHT.	SOLUTION.
Two angles A, B, and a side AC opposite to one of them, B.	8. The side BC opposite to the other given angle A.	Sin B : sin A : : sin A BC (7); the affect BC is uncertain, when it can be deter by this rule, that acc as A + B is greater than 180°, AC + also greater or less 180° (Sp. Ge. 10).
	9. The side AB adjacent to the given angles A, B.	From the unknown a draw CD perpendic AB; then R : cos A : : tan AC : t (c. 3); tan B : tan A AD : sin BD. BD biguous, and therefo = AD ± BD may four values, some of will be excluded b condition, that AB be less than 180°.
	10. The third angle ACB.	From the angle requi draw CD perpendic AB. R : cos AC : : tan A : co (c. 3); cos A : cos E ACD : sin BCD (8). affection of BCD is tain, and therefore A ACD ± BCD has values, some of whic be excluded by the tion that ACB is les 180°.

IVEN.	SOUGHT.	SOLUTION.
e three sides, 3, AC, and BC.	11. One of the angles A.	By proposition 13, any angle may be found when the three sides are given. To find A, sin AB · sin AC : R^2 = sin (S — AB) · sin (S — AC) : $\sin^2 \frac{1}{2}$ A. This is a convenient rule when expressed logarithmically.
e three ngles , B, C.	12. One of the sides BC.	Suppose the supplements of the three given angles A, B, C, to be *a*, *b*, *c*, and to be the sides of a spherical triangle. Find, by the last case, the angle of this triangle opposite to the side *a*, and it will be the supplement of the side of the given triangle opposite to the angle A, that is, of BC (Sp. Ge. 11); and therefore BC is found.

the foregoing table, the rules are given for ascertaining ffection of the arc or angle found, whenever it can be Most of these rules are contained in this one rule, h is of general application:—That when the part found her a tangent or a cosine, and of the tangents or cosines oyed in the computation of it, either one or three belong tuse angles, the angle found is also obtuse. This rule rticularly to be attended to in cases 5 and 7, where it ves part of the ambiguity.

hol.—The preceding rules are sufficient for the solution the cases of Spherical Trigonometry. There are various rules, however, which may be used in some cases with ntage; but the investigation of them, and also the exition of the preceding rule for determining the affection e part *sought, belong* properly to Analytical Trigono-r.

PROJECTIONS.

GENERAL DEFINITIONS.

1. The representation on a plane, of the important points and lines of an object, as they appear to the eye when situated in a particular position, is called the *projection* of the object.

2. The plane on which the delineation is made, is called the *plane of projection*, or *primitive*.

3. The point where the eye is situated, is called the *point of sight*, or the *projecting point*.

4. The point on the plane of projection, where a perpendicular to it from the point of sight meets the plane, is called its *centre*.

5. The line joining the point of sight and the centre, is called the *axis* of the primitive.

6. Any point, line, or other object to be projected, is called the *original* in reference to its projection.

7. A straight line drawn from the point of sight to any original point, is called a *projecting line*.

8. The surface, which contains the projecting lines of all the points of any original line, is called a *projecting surface*. When the original line is straight, the projecting surface will be a *projecting plane*.

Cor.—The projection of any point is the intersection of its projecting line with the primitive.

FIRST BOOK.

STEREOGRAPHIC PROJECTION OF THE SPHERE.

DEFINITIONS.

1. The *stereographic projection* of the sphere is that in which a great circle is assumed as the plane of projection, and one of its poles as the projecting point.

2. The great circle, upon whose plane the projection is made, is called the *primitive.*

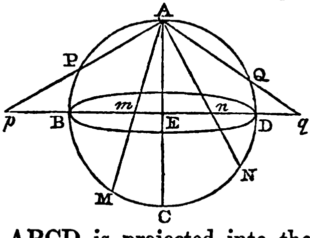
3. By the *semi-tangent* of an arc, is meant the tangent of half that arc.

4. By *the line of measures* of any circle of the sphere, is meant that diameter of the primitive, produced indefinitely, which is perpendicular to the line of common section of the circle and the primitive.

PROPOSITION I.

Every great circle which passes through the projecting point is projected into a straight line, passing through the centre of the primitive; and every arc of it, reckoned from the other pole of the primitive, is projected into its semi-tangent.

Let ABCD be a great circle, passing through A, C, the poles of the primitive, and intersecting it in the line of common section BED, E being the centre of the sphere. From A, the projecting point, let there be drawn straight lines AP, AM, AN, AQ, to any number of points P, M, N, Q, in the circle ABCD. These lines will intersect BED, which is in the same plane with them; let them meet it in the points p, m, n, q; then p, m, n, q, are the projections of P, M, N, Q. And thus the whole circle ABCD is projected into the straight line BED, passing through the centre of the primitive.



Again, because the points C and M are projected into E and m, the whole arc MC will be projected into the straight line mE, which, to the radius AE, is the tan mAE = tan $\frac{1}{2}$ MC. Thus, the arc MC is projected into its semi-tangent mE; PC into its semi-tangent pE, &c. All arcs, therefore, on the circle ABCD, reckoned from the pole C, are projected into their semi-tangents.

COR. 1.—Each of the quadrants contiguous to the projecting point is projected into an indefinite straight line, and each of those that are remote, into a radius of the primitive.

COR. 2.—Every small circle which passes through the projecting point is projected into that straight line which is its common section with the primitive.

COR. 3.—Every straight line, in the plane of the primitive, and produced indefinitely, is the projection of some circle on the sphere passing through the projecting point.

COR. 4.—The stereographic projection of any point in the surface of the sphere is distant from the centre of the primitive, by the semi-tangent of that point's distance from the pole opposite to the projecting point.

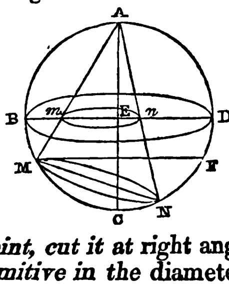
PROPOSITION II.

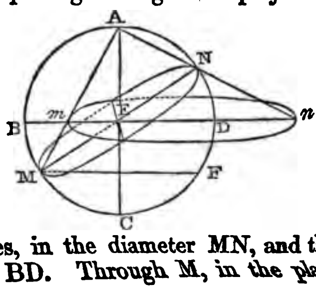
Every circle on the sphere which does not pass through the projecting point, is projected into a circle.

If the circle be parallel to the primitive, the proposition is evident.

For, a straight line, drawn from the projecting point to any point in the circumference, and made to revolve about the circle, describes the surface of a cone, which is cut by a plane (namely, the primitive), parallel to the base; and therefore the section (the figure into which the circle is projected) is a circle.

If the circle MN be not parallel to the primitive BD; let the great circle ABCD, passing through the projecting





point, cut it at right angles, in the diameter MN, and the primitive in the diameter BD. Through M, in the plane

of that great circle, let MF be drawn parallel to BD; let AM, AN, be joined, and meet BD in *m*, *n*. Then, because AB, AD, are quadrants, and BD, MF, parallel, the arc AM = AF, for BM = DF; since, if B and F were joined, the alternate angles would be equal; hence the angle AMF or *Amn* = ANM. Thus, the conic surface, described by the revolution of AM, about the circle MN, is cut by the primitive in a sub-contrary position (Conic Sections); therefore the section *mn* is, in this case, likewise a circle.

COR. 1.—The centres, and poles of all circles, parallel to the primitive, have their projections in its centre.

COR. 2.—The centre, and poles of every circle, inclined to the primitive, have their projections in the line of measures.

COR. 3.—All projected great circles cut the primitive in two points diametrically opposite; and every circle in the plane of projection, which passes through the extremities of a diameter of the primitive, or through the projections of two points that are diametrically opposite on the sphere, is the projection of some great circle.

For the original great circles cut the primitive in two points diametrically opposite.

COR. 4.—A tangent to any circle of the sphere, which does not pass through the projecting point, is projected into a tangent to that circle's projection; also, the circular projections of tangent circles touch one another.

COR. 5.—The extremities of the diameter, on the line of measures of any projected circle, are distant from the centre of the primitive, by the semi-tangents of the circle on the sphere's least and greatest distances from the pole opposite to the projecting point.

COR. 6.—The extremities of the diameter, on the line of measures of any projected great circle, are distant from the centre of the primitive, by the tangent and cotangent of half the complement of the great circle's inclination to the primitive.

For BM (second figure) measures the inclination of the circle MN to the primitive BD, and MC is its complement, *and angle MAC half* its complement. Also, since MAN is a

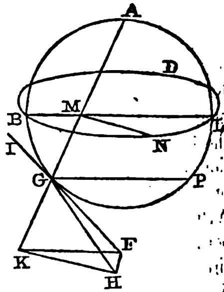
right angle, EAN is the complement of MAC. Also mE is the tangent of MAC, and En its cotangent.

COR. 7.—The radius of any projected circle is equal to half the sum, or half the difference, of the semi-tangents of the circle's least and greatest distances from the pole opposite to the projecting point, according as the circle does or does not encompass the axis of the primitive.

PROPOSITION III.

An angle, formed by two tangents, at the same point, in the surface of the sphere, is equal to the angle formed by their projections.

Let FGI and GH be the two tangents, and A the projecting point; let the plane AGF cut the sphere in the circle AGL, and the primitive in the line BML. Also, let MN be the line of common section, of the plane AGH, with the primitive. Then the angle FGH = LMN. If the plane FGH be parallel to the primitive BLD, the proposition is manifest. If not, through any point K, in AG produced, let the plane FKH, parallel to the primitive, be extended to meet FGH in the line FH. Then, because the plane AGF meets two parallel planes, BLD, FKH, the lines of common section, LM, FK, are parallel; therefore the angle AML = AKF. But, since A is the pole of BLD, the cords, and consequently the arcs AB, AL, are equal; and the arc ABG is the sum of the arcs AL, BG. Draw GP parallel to BL; then the arc BG = LP; for if B and P were joined, the alternate angles would be equal. Hence, the arc ABG = ALP, and the angle APG = AGP = AML = FKG. But angle APG = AGI (Pl. Ge. III. 32) = FGK. Consequently the angle FGK = FKG, and the side FG = FK. In like manner, HG = HK. Hence the triangles GHF, KHF, are equal in every respect, and the angle FGH = FKH = LMN.



COR. 1.—An angle, contained by any two circles of the sphere, is equal to the angle formed by their projections.

For, the tangents to these circles on the sphere are projected into straight lines, which either coincide with, or are tangents to, their projections on the primitive.

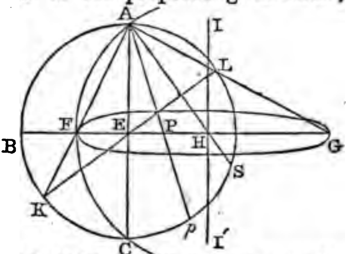
COR. 2.—An angle, contained by any two circles of the sphere, is equal to the angle formed by the radii of their projections, at the point of concourse.

When one of the given projected circles is a diameter of the primitive, for its radius a line perpendicular to it must be taken.

PROPOSITION IV.

The centre of a great circle's projection is distant from the centre of the primitive by the tangent of the great circle's inclination to the primitive, and its radius is the secant of the same.

Let A be the projecting point, ABC a great circle passing through it, perpendicular to the proposed great circle, KEL their line of common section, and BED the line of common section of ABCD, and the primitive. Then, because ABC is perpendicular both to the proposed great circle, and to the primitive, it is perpendicular to their line of common section, and consequently BE, EK, are likewise perpendicular to the same.



Hence BEK is the angle of inclination of the proposed circle to the primitive. Let AK, AL, be drawn, and meet BG in F, G; the straight line FG is the diameter of the projection. Let it be bisected in H, and let A, H, be joined. Because FAG is a right angle, HA = HF (IV. Cor. 5), and the angle HAF = HFA = FEK + FKE; *from these equals, taking the equal angles EAF, FKE, there remains* HAE = FEK, the angle of in-

clination. And, in the right-angled triangle AEH, to the radius AE, EH, is the tangent, and AH the secant of HAE.

COR. 1.—All circles which pass through the points A, C, are the projections of great circles, and have their centres in the line BG. All circles which pass through the points F, G, are the projections of great circles, and have their centres in the line II′, perpendicular to BG.

For, considering ABC as the primitive, any circle through A, C, is the projection of a great circle (St. Pr. II. Cor. 3).

Let AFC be an arc of a circle described on FG as a diameter; then, because HA = HF, as was proved, the circle cuts the primitive in A, C, the extremities of a diameter. Let IHI′ be perpendicular to FG, then, if an arc of a circle having its centre in II′ pass through F, it must pass through G. Also the points K, L, of its intersection with the primitive, are diametrically opposite. For $FE \cdot EG = AE^2 = KE \cdot EL$; and hence KEL is a diameter, and KFL is the projection of a great circle.

COR. 2.—It appears from this that II′ is the locus of the centres of the projections of all great circles that pass through the point F.

COR. 3.—BK is the measure of the inclination of the great circle to the primitive, and CS = 2 BK.

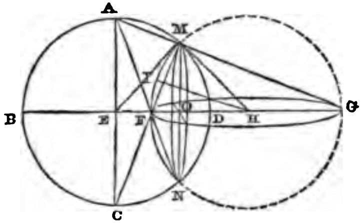
For HAE = BEK; therefore CS = 2 BK.

PROPOSITION V.

The centre of projection of a small circle, perpendicular to the primitive, is distant from the centre of the primitive, the secant of the circle's distance from its nearer pole, and the radius of projection, is the tangent of the same.

Let ABCD be a great circle, passing through the projecting point, and perpendicular to the proposed small circle, MON their line of common section, and BOD the line of common section of ABCD, and the primitive. Then BD is the axis, and O the centre of the small circle. Let AM and AN be drawn to meet BD in G, F; FG is the diameter on the line of measures of the small circle's projection. Let it be bisected in H, and EM, MH, joined. Then, because AE : EF = NO : OF, CE : EF = MO : OF; therefore the points C, F, M, are in a straight line, and that straight

line is perpendicular to AG. Hence HM = HF = HG. Therefore angle G = HMG. But EAM = EMA; hence EMA + HMG = EAM + G = a right angle; and hence EMH must be a right angle. Hence HM, the radius of projection, is the tangent of MD, the distance of the small circle from its nearest pole, and HE is the secant of the same.



Cor. 1.—EMH is a right angle, and EM, MH, are tangents to the two circles MFN and AMN.

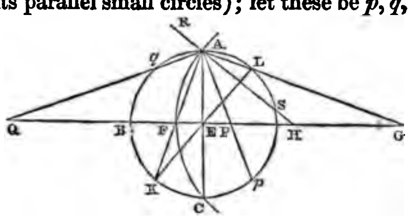
Cor. 2.—Any radius HI is a tangent to the great circle through the points B, I, D.

For (St. Pr. 3) the two circles being perpendicular, so are their projections, and hence also their radii at the point of intersection I.

PROPOSITION VI.

The projections of the poles of any circle, inclined to the primitive, are in the line of measures distant from the centre of the primitive, the tangent and cotangent of half its inclination.

Because ABCD is perpendicular to the plane of the great circle KL (as in Prop. 4), it passes through its poles (which are also the poles of all its parallel small circles); let these be p, q, and let Ap, Aq, meet BD in P, Q, their projections. Then the quadrants pK, CB, being equal, and CK common to both, pC will be equal to BK, which measures the inclination of the great circle (or its parallel small circles) to the primitive. Now, EP to the radius AE, is the tangent of $\frac{1}{2}$ pC, and EQ the tangent of $\frac{1}{2}$ qC, or cotangent of $\frac{1}{2}$ pC.



COR. 1.—The projection of that pole which is adjacent to the projecting point, is without the primitive, and the projection of the other within.

COR. 2.—The distances of either projected pole from the centres of the primitive and projected great circle, are directly proportional to the radii of these circles.

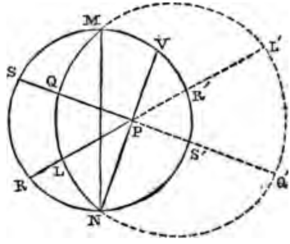
For AP bisects the angle EAH (Pr. I. 4, Cor. 3), and consequently AQ bisects the external angle EAR. Hence EP : PH = EA : AH, and EQ : QH = EA : AH (Pl. Ge. VI. 3).

Schol.—The projection of a circle perpendicular to the primitive is a diameter of it, and its poles are in the extremities of another diameter perpendicular to the former.

PROPOSITION VII.

If, from either pole of a projected great circle, two straight lines be drawn to meet the primitive and the projection, they will intercept corresponding arcs of these circles.

From the pole P, of the projected great circle MLN, let there be drawn any two straight lines PL, PQ, meeting the primitive in R, S, and the projection in L, Q; then shall the arc LQ be the projection of an arc equal to RS.



For SS′, RR′, are the projections of two circles (Pr. I. 1, Cor. 2), each of which passes through a pole of the primitive, and a pole of the great circle, and which therefore intercept equal arcs upon them (Sp. Ge. 18). Now, RS is one of the intercepted arcs, and the other is projected into LQ. Hence LQ, RS, are corresponding arcs.

COR.—Hence, if, from the point where the projections of two great circles intersect one another, two straight lines be drawn through their adjacent poles, these will intercept on the primitive an arc, which is the measure of their inclination.

For the point of intersection, being common to both circles, is at the distance of a quadrant from their poles; it is therefore the pole of an arc of a great circle passing through these

poles, and this arc on the sphere measures the inclination of the circles.

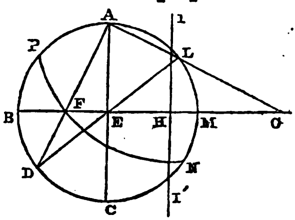
Schol.—The solutions of the following problems depend on the preceding principles :—

PROBLEM I.

To find the locus of the centres of the projections of all the great circles that pass through a given point.

Let F be any given point within the primitive.

Through F draw the diameter BM and AC perpendicular to it; draw AF and produce it to D; draw the diameter DL; draw AL, and produce it to meet BM in G; bisect FG perpendicularly by II′, and II′ is the required locus (Pr. I. 4, Cor. 2). Hence any circle PFN passing through F, and having its centre in any point as I in IHI′, is the projection of a great circle.



PROBLEM II.

Through any two points in the plane of the primitive, to describe the projection of a great circle.

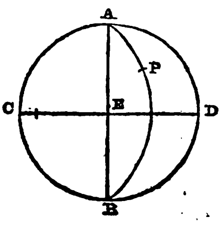
1. When one of the points is in the centre of the primitive.

Draw a diameter passing through the other point, and it will be the required projection. For the great circle passes through the pole of the primitive (Pr. I. 1, Cor. 3).

2. When one of the points is in the circumference, and the other is neither in the circumference nor in the centre.

Let A and P be the two points, and ACBD the primitive.

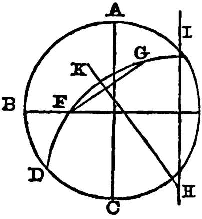
Draw the diameter AB, and describe the circle APB through the three points A, P, B; and it is the required circle (Pr. I. 2, Cor. 3).



3. When neither of the points is in the centre or circumference.

Let F, G, be the given points, and *ABC the primitive.*

Find IH the locus of the centres of all the projections of great circles passing through one of the points, as F (Pr. I. Prob. 1); join F, G, and bisect FG perpendicularly by KH; and the centre of every circle through F and G is in KH; but the centre of the required circle is in IH, hence H is its centre; and a circle DFG through the two given points described from the centre H, is the circle required.



PROBLEM III.

About some given point, as a pole, to describe the projection of a great circle.

1. When the given point is the centre of the primitive.

The required projection is evidently the primitive itself.

2. When the given point is in the circumference of the primitive.

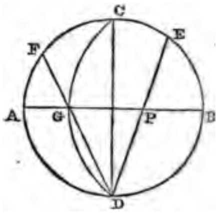
Draw a diameter through the given point, and another diameter perpendicular to the former; the latter diameter is the required projection.

For, since the primitive passes through the pole of the required projection, its original circle must pass through the pole of the primitive, and its projection is a diameter (Pr. I. 2, Cor. 3).

3. When the given point is neither in the centre nor the circumference of the primitive.

Let P be the given point, and ADBC the primitive.

Through P draw the diameter AB, and CD perpendicular to it. Draw DP, and produce it to E; make the arc EF equal to a quadrant; draw DF cutting AB in G; and the circle CGD through the points C, G, D, is the required circle.



For, considering APB as the primitive, and D its pole, *PG is evidently* the projection of a quadrant EF. Now, if *ADBC be the primitive*, since APB passes through P, the

pole of the required circle, it must pass through C, D, the poles of AB. Hence the required circle must pass through C, G, and D.

COR.—Hence the method of finding the pole of a projected great circle is evident.

1. When the projection is a diameter of the primitive, the extremities of the diameter perpendicular to it, are evidently its poles.

2. When the given projection is inclined to the primitive, as CGD.

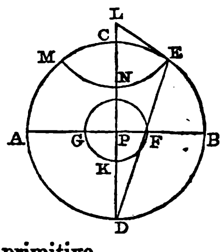
Join C, D, and draw the diameter AB perpendicular to CD. Draw DG, and produce it to F; make the arc FE a quadrant; draw DE, cutting AB in P, and P is the pole of the given circle.

PROBLEM IV.

To describe the projection of a small circle about some given point as a pole.

1. When the pole is in the centre of the primitive, or the original small circle parallel to the primitive.

Let AB, CD, be two perpendicular diameters of the primitive. Make CE equal to the distance of the small circle from its pole. Draw DE, cutting AB in F; from P as a centre, with the radius PF, describe the circle FGK; which will be the required projection.



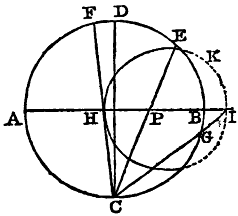
For PF is evidently the projection of CE, and the centre of the required circle is evidently in P.

2. When the given pole is in the circumference of the primitive, or the original circle is perpendicular to the primitive.

Let C be the given pole; AB, CD, two perpendicular diameters. Make CE equal to the distance of the circle from its pole. Draw EL a tangent to the primitive at E, and let it meet DC produced in L. A circle described from the centre L, with the radius LE, namely, MNE, is the required circle (*Pr. I. 5*).

3. When the pole is neither in the centre nor the circumference of the primitive.

Let P be the given point, and AB, CD, two perpendicular diameters of the primitive. Draw CP, and produce it to E; lay off EF, EG, each equal to the distance of the circle from its pole; draw CF, CG cutting AB in H and I, and on HI, as a diameter, describe the circle HKI, and it is the required projection. For if AB be the primitive, and C its pole; E the pole of a small circle, and F, G, two points in its circumference, then (Pr. I. 2, Cor. 5) HI is the diameter of its projection. Hence, if ACBD be the primitive, HI is evidently the diameter of the projected small circle, whose pole is P.



Cor.—The method of finding the projected pole of a given projected small circle is manifest from this problem.

1. When the small circle is concentric with the primitive, the centre of the latter is the projected pole of the former.

2. When the small circle is perpendicular to the primitive, as MNE, its pole is in C, the middle of the arc MCE.

3. When the circle is inclined to the primitive, as HKI, draw a diameter AB through its centre, and CD perpendicular to it; draw CH, CI cutting the primitive in F, G; bisect FEG in E; draw CE, and P is the required pole.

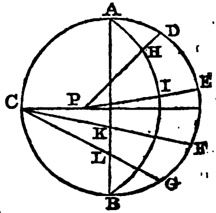
PROBLEM V.

To measure any given arc of a projected circle.

1. If the given arc be a part of the primitive, it may be measured as the arc of any other circle.

2. When the given arc is a part of a circle projected into a straight line.

Let KL be any given arc of the projected circle AB; find C its pole (Pr. I. Prob. 3, Cor.), and draw CK, CL cutting the primitive in F and G, and FG is the measure of KL (Pr. I. 7).



3. When the given circle is inclined to the primitive.

Let HI be the given arc of the projected circle AIB. Find P its pole; draw PH, PI cutting the primitive in D, E, and DE is the measure of HI.

PROBLEM VI.

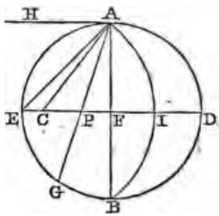
To measure the projection of a spherical angle.

1. When the circles containing the given angle are the primitive and a diameter of it.

The angle is a right angle (Pr. I. 1).

2. When one of the circles is the primitive, and the other is a circle inclined to it.

Let AEB be the primitive, and AIB the other circle, and IAD the angle. Find F and C their centres; draw AC, AF, and the angle CAF measures the given angle (Pr. I. 3, Cor. 2). Or find F and P their poles; draw AP, AF cutting the primitive in G and B, and GB measures the given angle (Pr. I. 7, Cor.)



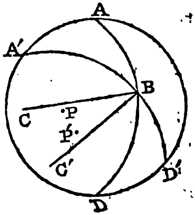
3. When one of the circles is a diameter of the primitive, and the other is inclined to the latter.

Let AFB and AIB be the two circles, and FAI the given angle.

Draw the radius AC of the circle AIB, and AH perpendicular to AFB, and the angle HAC measures the given angle (Pr. I. 3, Cor. 2). Or find P and E the poles of the circles; draw AE, AP, then GE measures the given angle.

4. When both the circles are inclined to the primitive.

Let ABD, A′BD′, be the two circles, and ABA′ the given angle. Find C, C′, the centres of the circles, then the two radii drawn from these to B, will contain an angle CBC′ equal to that at B. Or find P, P′, the poles of the circles, and lines drawn from B through these points, will intercept on the primitive an arc which measures the given angle.



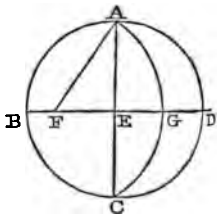
PROBLEM VII.

Through a given point in a given projected great circle, to describe the projection of another great circle, cutting the former at a given angle.

Let ABCD be the primitive, and Z the given angle.

1. When the given circle is the primitive.

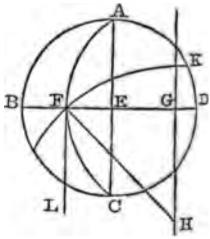
Let A be the given point; draw the perpendicular diameters AC, BD; make angle EAF = Z; and from F, as a centre, with a radius FA, describe the circle AGC; it is the required projection (Pr. I. 3, Cor. 2).



When the angle is a right angle, the diameter AC is evidently the required projection.

2. When the given projected circle is a diameter of the primitive.

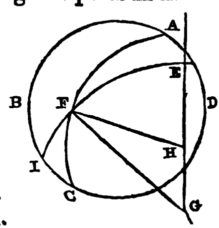
Let BD be the given projection, and F the given point. Find GH the locus of all the great circles passing through F (Pr. I. Prob. 1); draw IL perpendicular to BD, and FH making an angle LFH = Z; from the centre H, with the radius HF, describe the circle IFK; it is the required projection (Pr. I. 3, Cor. 2).



If the angle be a right angle, G is the centre, and AFC the required projection, for angle LFG = a right angle. Or, since the required circle is, in this case, perpendicular to BFD, it must pass through its poles A and C. Hence the circle A, F, C, passing through the three points A, F, C, is the required projection.

3. When the given circle is inclined to the primitive.

Let AFC be the given circle, and F the given point in it. Find EG the locus of the centres of all the great circles passing through F (Pr. I.



Prob. 1). Draw FH a radius of the given circle, and draw FG, making the angle GFH = Z; from the centre G, with the radius GC, describe IFE; and it is the required projection (Pr. I. 3, Cor. 2).

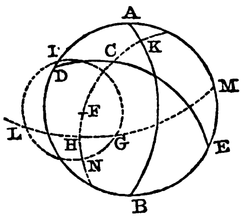
When the angle Z is a right angle, draw from F a line perpendicular to FH, and it will cut EG in the centre of the required circle. Or since, in this case, the required projection must pass through the pole of AFC; find its pole, and describe the projection of a great circle passing through this pole and the point F (Pr. I. Prob. 2), and it will be the required circle.

PROBLEM VIII.

Through a given point in the plane of the primitive, to describe the projection of a great circle cutting that of another great circle at a given angle.

Let AKB be the given circle, Z the given angle, and C the given point in the plane of the primitive AMB.

Find F the pole of AKB, and about it describe a small circle IGN, at a distance from its pole equal to the measure of angle Z. About the given point C, as a pole, describe a great circle LHM, intersecting the small circle in L and G. About either of these points, as G, for a pole, describe a great circle DCE, and it is the required projection. For the circle DCE must pass through C, since C is at the distance of a quadrant from G, a point of the circle LGM. Also, the distance between F and G, the poles of AKB and DCE, is the measure of the given angle, and hence the inclination of the circles is equal to that angle.



Schol. 1.—Let an arc of a great circle FCK be described through F and C; then, FK and CH being quadrants, FH = CK. Now, FH must not exceed FN the measure of the angle, otherwise the circle LHM would not meet IGN, and the problem would be impossible. But CK = FH; therefore the distance of the given point from the given *circle must not* exceed the measure of the angle.

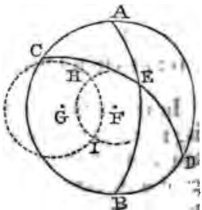
Schol. 2.—If the point C were in the centre of the primitive, the circle LGM would coincide with the primitive. If C were in the circumference of the primitive, the circle LGM would be a diameter perpendicular to that passing through C.

PROBLEM IX.

To describe the projection of a great circle that shall cut the primitive and a given great circle at given angles.

Let ADB be the primitive, AEB the given circle, and X, Y, the given angles, which the required circle makes respectively with these circles.

About F, the pole of the primitive, describe a small circle at a distance equal to the measure of angle X, and about G, the pole of AEB, describe another small circle at a distance equal to the measure of angle Y. Then from either of the points of intersection H, I, as I for a pole, describe the great circle CED, and it is the required circle. For the distances of its pole I from F and G, the poles of the given circles, are equal to the measures of the angles X and Y; and therefore the inclinations of CED to the given circles are equal to these angles.



Schol.—When any of the angles exceeds a right angle, the distance of the small circle from its pole is greater than a quadrant. The same small circle will be determined by finding the more remote pole, that is, the projection of the pole nearest to the projecting point, and then describing a small circle about it at a distance equal to the supplement of the measure of the angle.

EXERCISES.

1. If, from one of the points, in which a perpendicular small circle meets the primitive, a straight line be drawn to any point in the circumference of its projection, and continued to meet the diameter of the primitive that is perpendicular to the line of measures, the point of concourse will be the pole of the projected great circle, which passes through *the poles of* the small circle, and that point in the circumference of its projection.

2. If two equal circles, one of which is parallel, and the other inclined to the primitive, be projected, the distances of the pole of the inclined projected circle, from the centres of the projections, will be directly proportional to the radii of these projected circles.

3. If two equal circles, one of which is parallel, and the other inclined to the primitive, be projected, straight lines, drawn through the pole of the inclined projected circle, will intercept corresponding arcs on the projection.

SECOND BOOK.

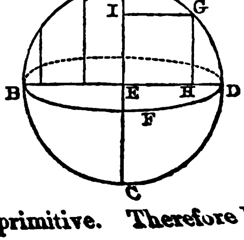
ORTHOGRAPHIC PROJECTION OF THE SPHERE.

In the *orthographic projection* of the sphere, the projecting point is still supposed to be in the axis of a great circle, assumed as the primitive or plane of projection; but at so great a distance, that a straight line drawn from it to any point of the sphere, may be considered as perpendicular to the plane of the primitive.

The orthographic projection of any point, therefore, is where a perpendicular from that point meets the primitive.

PROPOSITION I.

Every great circle, perpendicular to the primitive, is projected into a diameter of the primitive; and every arc of it, reckoned from the pole of the primitive, is projected into its sine.



Let BFD be the primitive, and ABCD a great circle perpendicular to it, passing through its poles A, C; then the diameter BED, which is their line of common section, will be the projection of the circle ABCD.

For, if from any point as G, in the circle ABC, a perpendicular GH fall upon BD, it will also be *perpendicular to the plane* of the primitive. Therefore H

is the projection of G. Hence the whole circle is projected into BD, and any arc AG into EH = GI, its sine.

COR. 1.—Every arc of the great circle, reckoned from its intersection with the primitive, is projected into its versed sine.

COR. 2.—The orthographic projection of any point on the surface of the sphere, is, within the primitive, distant from its centre, by the sine of that point's distance from either pole of the primitive.

COR. 3.—Every small circle, perpendicular to the primitive, is projected into its line of common section with the primitive, which is also its own diameter; and every arc of the semicircle above the primitive, reckoned from the middle point, is projected into its sine.

COR. 4.—Every diameter of the primitive is the projection of a great circle, and every other cord the projection of a small circle.

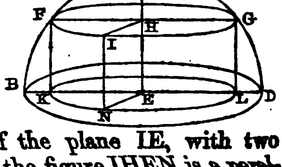
COR. 5.—A straight line, perpendicular to the primitive, is projected into a point; a parallel to the primitive, into an equal line; and one inclined to the primitive, into a less line, such that the radius is to the cosine of the inclination, as the inclined line to its projection.

COR. 6.—A spherical angle, at the pole of the primitive, also any rectilineal angle, whose plane is parallel to the primitive, is projected into an equal angle.

PROPOSITION II.

A circle parallel to the primitive is projected into a circle equal to itself, and concentric with the primitive.

Let the small circle FIG be parallel to the plane of the primitive BND. The straight line HE, which joins their centres, is perpendicular to the primitive; therefore E is the projection of H. Let any radius HI and IN perpendicular to the primitive be drawn. Then IN, HE, being parallel, are in the same plane; therefore IH, NE,



the lines of common section of the plane IE, with two parallel planes, are parallel, and the figure IHEN is a paral-

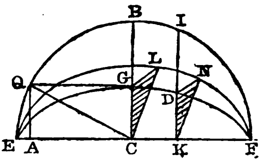
lelogram. Hence NE = IH, and consequently FIG is projected into an equal circle KNL, whose centre is E.

COR.—The radius of the projection is the cosine of the parallel's distance from the primitive, or the sine of its distance from the pole of the primitive.

PROPOSITION III.

An inclined circle is projected into an ellipse, whose transverse axis is the diameter of the circle.

Case 1. Let ELF be a great circle, inclined to the primitive EBF, and EF their line of common section. From the centre C, and any other point K, in EF, let the perpendiculars CB, KI, be raised in the plane of the primitive, and CL, KN, in the plane of the great circle, meeting the circumference in L, N. Let LG, ND, be perpendicular to CB, KI; then G, D, are the projections of L, N. And because the triangles LCG, NKD, are equiangular, $CL^2 : CG^2 = NK^2 : DK^2$, or $EC^2 : CG^2 = EKF : DK^2$; therefore the points G, D, are in the curve of an ellipse, of which EF is the transverse, and CG the semi-conjugate axis.



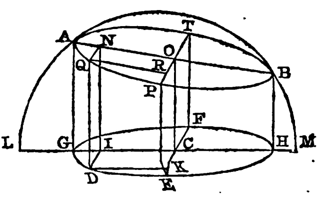
COR. 1.—In a projected great circle, the semi-conjugate axis is the cosine of the great circle's inclination to the primitive.

COR. 2.—Perpendiculars to the transverse axis intercept corresponding arcs of the projection and the primitive.

COR. 3.—The eccentricity of the projection is the sine of the great circle's inclination to the primitive.

For $LG^2 = LC^2 - CG^2 = EC^2 - CG^2$; LG is therefore equal to the eccentricity, and it is also = Sin LCG.

Case 2. Let AQB be a small circle, inclined to the primitive, and let the great circle LBM, perpendicular to both, intersect them in the lines AB, LM. From the centre O, and any other point N, in the diameter



AB, let the perpendiculars TOP, NQ, be drawn in the plane of the small circle, to meet its circumference in T, P, Q. Also, from the points A, N, O, B, let AG, NI, OC, BH, be drawn perpendicular to LM, and from P, Q, T, PE, QD, TF, perpendicular to the primitive; then G, I, C, H, E, D, F, are the projections of these points. Because OP is perpendicular to LBM, and OC, PE, being perpendicular to the primitive, are in the same plane, the plane COPE is perpendicular to LBM. But the primitive is perpendicular to LBM. Therefore the line of common section EC is perpendicular to LBM, and to LM. Hence CP is a parallelogram, and $EC = OP$. In like manner, FC, DI, are proved perpendicular to LM, and equal to OT, NQ. Thus, ECF is a straight line, and equal to the diameter PT or AB. Let QR, DK, be parallel to AB, LM; then $RO = NQ = DI = KC$, and $PR \cdot RT = EK \cdot KF$. But $AO : CG = NO : CI$; therefore $AO^2 : CG^2 = QR^2 : DK^2$, or $EC^2 : CG^2 = EK \cdot KF : DK^2$.

COR. 1.—The transverse axis is to the conjugate, as radius to the cosine of the circle's inclination to the primitive.

COR. 2.—Half the transverse axis is the cosine of half the sum of the greatest and least distances of the small circle from the primitive.

COR. 3.—The extremities of the conjugate axis are in the line of measures distant from the centre of the primitive, by the cosines of the greatest and least distances of the small circle from the primitive.

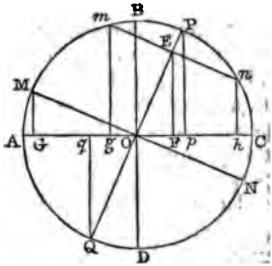
COR. 4.—If, from the extremities of the conjugate axis of any elliptical projection, perpendiculars be raised (in the same direction, if the circle do not intersect the primitive, but, if otherwise, in opposite directions), they will intercept an arc of the primitive, whose cord is equal to the circle's diameter.

PROPOSITION IV.

The projected poles of an inclined circle are, in its line of measures, distant from the centre of the primitive, the *sine of the circle's* inclination to the primitive.

Let ABCD be a great circle, perpendicular, both to the

primitive and the inclined circle, and intersecting them in the diameters AC, MN. Then ABCD passes through the poles of the inclined circle; let these be P, Q, and let Pp, Qq, be perpendicular to AC; p, q, are the projected poles, and it is evident that pO = Sin BP, or sin MA, the inclination.



COR. 1.—The centre of the primitive, the centre of projection, the projected poles, and the extremities of the conjugate axis, are all in one and the same straight line.

COR. 2.—As radius is to the sine of a small circle's inclination to the primitive, so is the cosine of its distance from its own pole to the distance of the centre of its projection from the centre of the primitive.

$$OM : MG = OE : OF.$$

EXERCISES.

1. To describe the projection of a small circle parallel to the primitive, its distance from the pole of the primitive being given.

2. To project an inclined circle, whose distance from its pole and inclination to the primitive, are given.

3. To find the poles of a given projection.

4. To measure any part of a given projection.

5. In any inclined circle, two diameters, which cut each other at right angles, are projected into conjugate diameters of the ellipse.

6. In every elliptical projection, half the transverse axis is to the eccentricity, as radius to the sine of the circle's inclination to the primitive; and half the conjugate axis is to the eccentricity, as radius to the tangent of the same.

7. Through two given points, in the plane of the primitive, to describe the projection of a great circle.

8. Given the distance of a circle from its pole, and the projection of that pole, to describe the projection of the circle.

9. Through a given point, in the plane of the primitive, to describe the projection of a great circle, having a given inclination to the primitive.

THIRD BOOK.

PERPENDICULAR PROJECTION.

DEFINITIONS.

1. In Perpendicular Projection, the position of the point of sight is at an *indefinitely great* distance in the axis of the primitive.

2. The perpendicular projection of any point is called the *seat* of that point.

3. Two planes at right angles are sometimes assumed for planes of projection, and they are then called *co-ordinate planes.*

4. When the two co-ordinate planes are a horizontal and a vertical plane, the projections on them are respectively called the *horizontal* and *vertical projection.*

5. In the case of an object of a regular figure, such as a house, the horizontal projection is called the *plan;* and the vertical, the *elevation.* When the projection is that of a section, it is called a *section.*

Cor. 1.—A projecting line in perpendicular projection is a line drawn from the original point perpendicular to the primitive (Def. 1).

Cor. 2.—If the seats of a point on any two planes, not parallel, be given, the position of the point in space may be found; for it will be the point of intersection of the two projecting lines drawn from the two seats.

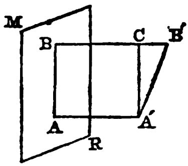
Cor. 3.—If the projections, on two planes, of all the remarkable points and lines of an object, be given, the figure and dimensions of the object may be determined.

This method of projection is employed in the construction of plans and sections of houses, the plans of philosophical apparatus, and of various objects in engineering and carpentry.

PROPOSITION I.

The projection of a line inclined to the primitive is less than the line in the ratio of radius to the cosine of its inclination.

Let A′B′ be a line inclined to the primitive MR, and AB its projection, and A′C parallel to AB, then A′B′ : AB = Radius : cosine B′A′C.



For, the projecting lines A′A, and B′B, being perpendicular to the primitive, are parallel; and the angles at A and B are right angles, and A′C is parallel to AB; therefore AC is a rectangle, and A′C = AB; and the inclination of A′B′ to A′C is the same as its inclination to AB or to the primitive. Also, A′B′ : A′C = Radius : cosine B′A′C; or A′B′ : AB = Radius : cosine of inclination.

Cor. 1.—If a line parallel to the primitive join any two projecting lines, and if a line on the primitive parallel to it join the same lines, these lines, with the intercepted portions of the projecting lines, form a rectangle.

For AC is a rectangle.

Cor. 2.—The projection of a line parallel to the primitive, is equal to the original.

For the projection of A′C is AB, and they are equal.

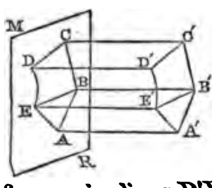
Cor. 3.—The projection of a line perpendicular to the primitive, is a point.

For the projection of the line B′C is the point B.

PROPOSITION II.

The projection of a plane figure parallel to the primitive, is a figure similar and equal to the original.

Let ABCDE be the projection of A′B′C′D′E′, which is parallel to the primitive, these figures are similar and equal.



For the sides AB, BC, &c. are equal to A′B′, B′C′, &c. (Pr. III. 1, Cor. 1). Also, if any plane perpendicular to the primitive cut the two figures in lines B′E′,

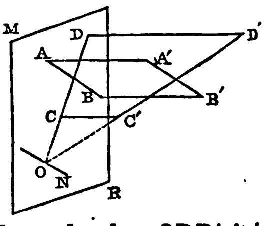
BE, these lines are corresponding dimensions of the two figures, and they are equal. Also, any two lines in one of the figures, as A′B′, B′E′, have the same inclination as the corresponding lines of the other figure AB, BE.

COR.—The intersection of the primitive, with a plane perpendicular to it, is the projection of this plane's intersection with any other plane or surface or solid figure.

PROPOSITION III.

If a plane figure be inclined to the primitive, any of its dimensions parallel to the line of common section of its plane with the primitive, is equal to its projection; but any of its dimensions perpendicular to this line exceeds its projection in the ratio of radius to the cosine of obliquity.

Let ON be the intersection of a plane figure with the primitive MR; A′B′ a section of it by a projecting plane parallel to ON; and C′D′ a section of it by a projecting plane perpendicular to ON, cutting ON in O. Draw projecting lines from A′, B′, C′, and D′; and let AB be the projection of A′B′; CD of C′D′; and OC of OC′.



Then, since ON is perpendicular to the plane ODD′, it is so to the lines OD, OD′; and therefore, the inclination of the plane of the figure with the primitive is measured by the angle DOD′. Now (Pr. III. 1) OD′ : OD = Radius : cosine DOD′; but OD′ : OD = C′D′ : CD. Therefore C′D′ : CD = Radius : cosine DOD′.

Again, since A′B′ is parallel to AB, AB′ is a rectangle, and AB = A′B′.

COR. 1.—The projection of a plane figure inclined to the primitive, is less than the original in the ratio of the radius to the cosine of obliquity.

For every line perpendicular to the line of common section exceeds its projection in this ratio; and the lines parallel to the line of common section are equal to their projections; hence the whole figure is greater than its projection in this ratio.

Cor. 2.—The projection of a circle inclined to the primitive is an ellipse, unless it be a subcontrary section, in which case it is a circle.

For, when its projection is a subcontrary section of the cylindrical projecting surface, it is a circle (Conic Sections, IV. 2).

Again, when the circle is inclined, its diameter, which is parallel to the line of common section of its plane with the primitive, is projected into an equal line; and all the dimensions perpendicular to this line are projected into less lines in the ratio of radius to the cosine of obliquity. Hence the projection is an ellipse, of which the transverse axis is equal to the diameter of the circle, and its conjugate less in the preceding ratio.

Cor. 3.—The projection of a sphere is a circle of equal diameter.

FOURTH BOOK.

PERSPECTIVE.

DEFINITIONS.

1. The theory of *Linear Perspective* treats of the method of projecting objects on a vertical plane from some given point of sight.

2. The plane of projection is also called the *perspective plane*, or the plane of the *picture.*

3. The point of sight is also called the *point of view.*

4. The centre of the perspective plane is also called the *centre of the picture.*

5. The distance of the point of view from the primitive, is called the *distance of the picture.*

6. A vertical plane passing through the axis of the primitive, is called the *vertical plane;* and a horizontal plane passing through it, is called the *horizontal plane.*

7. The intersection of the primitive with the horizontal plane, is called the *horizontal line;* and its intersection with the vertical plane is called the *vertical line.*

8. The intersection of the primitive with the ground plane, or that of the sensible horizon, is called the ground line.

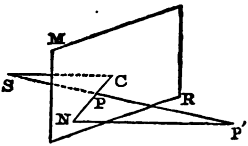
9. The intersection of the primitive with a line from the point of view parallel to any original line, is called the *vanishing point* of that line.

10. Two points on the horizontal line, whose distances from the centre are equal to the distance of the picture, are called *points of distance.*

PROPOSITION I.

If the distance between the centre and the seat of any point be cut in the ratio of the distance of the picture to the distance of the point from the primitive, the point of section will be the perspective of the point.

Let MR be the primitive, C its centre, S the point of sight, P′ the given point, and N its seat. Then, if the point P in NC be taken so that CP : PN = SC : P′N, P is the perspective of P′.



For SC and NP′ are parallel, and therefore in the same plane, and they are in the same plane with CN. Also, the angles C and N are equal, being right angles, and SC : CP = P′N : NP; and hence the triangles SCP and P′NP are similar. Therefore the angles at P are equal, and therefore SP, PP′, are in one straight line; and P is therefore the perspective of P′.

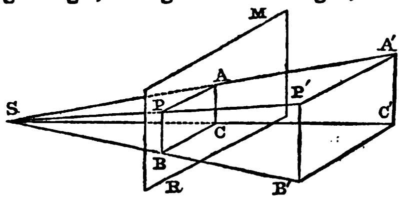
PROPOSITION II.

The sum of the distance of the picture, and the distance of any point from it, is to the distance of the picture, as the distance of that point from the vertical plane to the distance of its projection from the vertical line; and also as the distance of that point from the horizontal plane to the distance of its projection from the horizontal line.

Let MR be the primitive, P′ the given point, P its projection, and S the point of sight, and C the centre.

Through P′ let a plane P′C′ pass parallel to MR, cutting the horizontal and vertical planes in the lines B′C′ and C′A′, and in that plane draw P′A′ and P′B′ parallel to the opposite sides of the figure P′C′; join S and the points B′, P′, A′ and let these lines cut the primitive in B, P, A.

Because the opposite sides of the figure P′C′ are parallel, and the angle C′ a right angle, the figure is a rectangle; and PC being a section of the prism SA′B′ by a plane parallel to the base, it is similar to the base, and is therefore a rectangle. From similar triangles SC′:



SC = A′C′ : AC = P′B′ : PB; and SC′ : SC = B′C′ : BC = P′A′ : PA; where P′B′ and PB are the distances of the point and its perspective from the horizontal plane, and P′A′, PA, their distances from the vertical plane.

COR.—Let the distances of the perspective of the point from the horizontal and vertical planes be respectively v and h, and those of the point itself v' and h', the distance of the picture d, and that of the point from the primitive p, then

$$d + p : d = v' : v, \text{ and } v = \frac{dv'}{d+p};$$

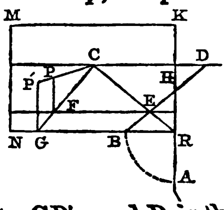
$$\text{also, } d + p : d = h' : h, \text{ and } h = \frac{dh'}{d+p}$$

PROPOSITION III.

To find the perspective of a point by means of a plane construction.

Let MR be the primitive, C its centre, CD the horizontal line, D the point of distance, and GR the ground line.

Produce the side KR, and make RA = p, the point's distance beyond the plane, RG its distance from the vertical plane passing through KR, and RH its distance above the ground plane.



Make RB = RA; and join C, R, and D, B; through E draw EF parallel to NR; join C, G; make P′G perpendicular to NR and = RH; join C, P′; and draw FP parallel to GP′; and P is the perspective of the point.

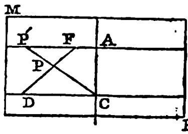
For P′ is evidently the seat of the point; and CP : PP′ = CF : FG = CE : ER = CD : BR = $d : p$. Hence (Pr. IV. 1) P is the perspective of the point.

Cor. 1.—It is evident that the line EF is the locus of the perspective of all the points in the line drawn on the ground plane parallel to the ground line, and at a distance equal to RA from the primitive.

Cor. 2.—The point F is evidently the perspective of the seat of the given point on the ground plane.

Schol.—The lines RN, RK, are sometimes called the *scale of the front* and the *scale of heights*, and CR is called the *flying scale.*

Cor. 3.—The perspective of the point may be found more simply thus:—Let CA be the distance of the point from the horizontal plane, and AP′ its distance from the vertical plane, D the point of distance, AP′ parallel to DC, and CA perpendicular to it. Make P′F equal to the distance of the point from the primitive; join P′, C, and D, F, then P is the perspective of P′. For P′ is evidently the seat of the given point, and CP : PP′ = CD : FP′; and therefore P is the perspective of the given point (Pr. IV. 1).



PROPOSITION IV.

The perspective of a plane figure parallel to the primitive is a similar figure, the dimensions of which are to the corresponding dimensions of the given figure in the ratio of the distance of the picture to the sum of the distance of the picture, and the distance of the point from the primitive.

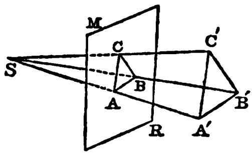
Let S be the point of sight, and let the figure A′B′C′ be parallel to MR, then it is evident (So. Ge. II. 14, Cor. 1) that ABC is similar to A′B′C′; and that AC : A′C′ = SC : SC′ = $d : p + d$.

Cor. 1.—The projection of a straight line is a straight line, unless it be directed to the eye, in which case it is a point.

The projecting surface for any straight line as A′B′ is

evidently a plane SA′B′, the intersection AB of which with the primitive is a straight line.

When the line, as AD, is directed to S, its projection is the point A.



COR. 2.—The perspective of a straight line parallel to the primitive is parallel to the original.

COR. 3.—The perspective of a plane figure, whose plane passes through the point of sight, is a straight line.

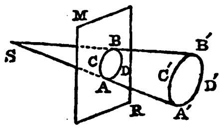
For the projecting surfaces of its boundaries evidently lie in one plane.

PROPOSITION V.

The projection of a circle inclined to the primitive is an ellipse, unless it be a subcontrary section of the projecting conical surface.

The perspective ACBD of the inclined circle A′C′B′D′ is an ellipse, unless it be a subcontrary section.

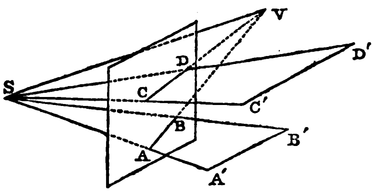
For ACBD is a section of the projecting conical surface SA′B′, and it is therefore an ellipse, unless it be a subcontrary section, in which case it is a circle (Conic Sections IV. 3).



PROPOSITION VI.

The perspectives of parallel lines converge towards their vanishing point.

Let A′B′ and C′D′ be two parallel lines, and V their vanishing point, their projections AB and CD converge towards V.



For since SV and A′B′ are parallel, the projecting plane of A′B′ will pass through SV, and therefore through V; and hence the perspective of A′B′ passes

through V. Similarly it is shown that the perspective of C′D′ passes through V.

COR. 1.—The perspectives of lines perpendicular to the primitive converge towards its centre.

For the centre is their vanishing point.

COR. 2.—The perspectives of lines parallel to the primitive are parallel.

For the vanishing point in this case is infinitely distant. It is evident also from Cor. 2 to Prop. IV.

COR. 3.—The perspectives of parallel horizontal lines converge towards a point in the horizontal line.

COR. 4.—The perspectives of parallel lines that are also parallel to the vertical plane, converge towards a point in the vertical line.

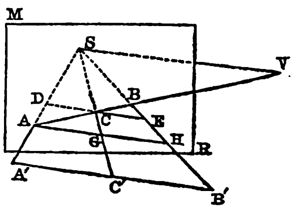
COR 5.—The perspectives of horizontal lines, whose inclination to the primitive is half a right angle, converge towards the point of distance.

PROPOSITION VI.

Given the perspective of a straight line and its vanishing point, to find a point in it, which is the perspective of a point in the original line, that divides it in a given ratio.

Let AB be the given perspective, V the vanishing point, and P, Q, two lines in the given ratio.

From V draw any line VS in the plane of the primitive, and take any point S in it; join S, A, and S, B, and produce these lines. Through A draw AH parallel to SV, and divide AH in G, so that AG : GH = P : Q; join S, G, and the point of intersection C is the point required.



For through C draw DE parallel to SV; then, from similar triangles AC : AV = DC : SV, and AC · SV = AV · DC. Also CB : BV = CE : SV, and CB · SV = BV · CE. Also DC : CE = P : Q, and AV : BV = AV : BV; therefore (Pl. Ge. VI. 23, Cor. 1) AV · DC : BV · CE = P · AV : Q · BV. But AC · SV :

$CB \cdot SV = AV \cdot DC : BV \cdot CE$; and therefore $AC \cdot SV : CB \cdot SV = P \cdot AV : Q \cdot BV$; or $AC : CB = P \cdot AV : Q \cdot BV$.

Let X and Y be the sides of two squares respectively equal to the given rectangles $P \cdot AV$ and $Q \cdot BV$, and let Z be a third proportional to X and Y. Then $AC : CB = X^2 : Y^2 = X : Z$. Hence divide AB in C in the ratio of X to Z.

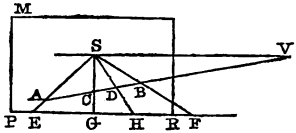
The position of C, thus determined, is independent of the length and direction of SV; and therefore SV may be drawn of any length and in any plane. Hence let S be the point of sight on the farther side of the primitive MR, and SV will then be parallel to the original line, the extremities of which will lie in SA and SB produced. Let A′B′ be the original line, then DE and AH are parallel to A′B′ or SV, and these lines, with the given line AB, lie in one plane. Divide A′B′ in C′, so that $A'C' : C'B' = P : Q$; then since $DC : CE = P : Q$, if SC and CC′ be drawn, they will lie in one straight line, and therefore C is the perspective of C′.

PROPOSITION VII.

Given the perspective of a straight line which is divided into segments having a given ratio and its vanishing point; to find those segments of its perspective, that are respectively the perspectives of the segments of the original line.

Let AB be the given perspective, and V the vanishing point, and X, Y, Z, three lines in the ratio of three segments into which the original line is divided.

Draw VS parallel to the ground line PR, and take in it any convenient point S; draw SA, SB, and produce them to E and F in PR. Divide EF in G and H similarly to the original line; join S, G, and S, H; then AC, CD, DB, are the required segments.



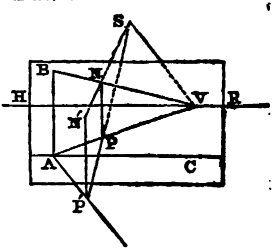
For (Pr. IV. 6), since SV and EF are parallel, and EF is similarly divided to the original line; therefore C, D, are the perspectives of the points of section, and consequently *AC, CD, DH*, are those of the segments of the original line.

PROPOSITION VIII.

To find the perspective of a vertical line of a given length, having given the perspective of its base and the horizontal line in which the seat of its base is situated.

Let P be the perspective of its base, and AC the line in which its seat is.

In the horizontal line HR take any point V; draw VP, and produce it to cut AC in A; draw the vertical line AB equal to the given line; join V, B, and from P draw PN parallel to AB, and PN will be the required perspective.



For, join the point of sight S and V, and draw AP′ parallel to SV, and AP′ will therefore lie in a horizontal plane containing the base of the given line, and it will also lie in the same plane with SV and AV. Join SP, and produce it to cut AP′ in P′, which is the base of the given line, since it must lie in the line SP, and also in the plane P′AC. From P′ draw a vertical line P′N′ to meet SN produced in N′. Then $P'N' : PN = SP' : SP = VA : VP = AB : PN$; and therefore $AB = P'N'$, and P′N′ is the given line, and therefore PN is its perspective.

CONIC SECTIONS.

FIRST BOOK.

PARABOLA.

DEFINITIONS.

1. If a straight line and a point be given in position, the locus of a point which is equally distant from them, is a curve called a *parabola*.

2. The given line is named the *directrix*, and the given point the *focus*.

3. The *vertex* of the parabola is the middle of the perpen-

dicular, which falls upon the directrix from the focus: and the *axis*, or *principal diameter*, is that part of the perpendicular produced indefinitely which falls within the curve.

4. Any straight line, drawn from a point in the curve, parallel to the axis, and in the same direction, is called a *diameter*, and the point in the curve its *vertex*.

5. An *ordinate* to a diameter is a straight line, terminated on both sides by the curve, and bisected by that diameter: the part of the diameter which it cuts off, is called an *absciss*.

6. The *parameter* of a diameter is four times the distance of its vertex from the directrix.

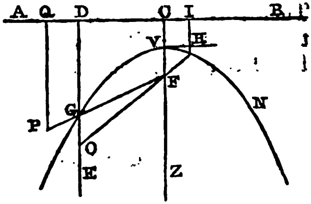
7. A *tangent* is a straight line, which meets the curve in one point only, and every where else falls without it.

8. A *subtangent* is that part of a diameter which is intercepted by a tangent, and an ordinate to that diameter from the point of contact.

PROPOSITION I.

The distance of a point from the focus is greater than its distance from the directrix, if the point be without the parabola, but less if it be within.

Let GVN be a parabola, whose directrix is AB, vertex V, and focus F; and let P be a point without the curve, that is, on the same side of the curve with the directrix: then, if PF be joined, and PQ be drawn perpendicular to AB, PF will be greater than PQ.



For, as PF necessarily cuts the curve, let G be the point of section, GD perpendicular to AB, and P, D, joined. Then, because GF = GD (I. Def. 1), PF = PG + GD. Hence PF is greater than PD (Pl. Ge. I. 20), and consequently still greater than PQ (Pl. Ge. I. 19).

Again, let O be a point within the curve. The perpendicular OD, upon AB, necessarily cuts the curve; let G be the point of section, and let GF, FO, be joined. Then OD is equal to the sum of OG, GF, and therefore greater than OF (Pl. Ge. I. 20).

COR. 1.—A point is without, in, or within, the curve, according as its distance from the focus is greater, equal, or less, than its distance from the directrix.

COR. 2.—A perpendicular to the axis, at its vertex, is a tangent to the parabola.

For HI = VC = VF, and VF ∠ FH, therefore HI ∠ FH; and hence (Cor. 1) H is without the curve. In a similar manner, every other point of VH may be shown to be without the curve; therefore VH is a tangent.

PROPOSITION II.

Every straight line perpendicular to the directrix meets the parabola, and every diameter falls wholly within it.

Let DG be perpendicular to the directrix, at any point of it, then shall DG meet the curve (figure to next proposition).

For, DF being joined, let the angle DFG be made equal to GDF, and let FG meet DG (Pl. Ge. I. 29, Cor.), which is parallel to FC, in G. The triangle DGF, having the angles at D and F equal, will also have the sides GD, GF, equal; and therefore G is a point in the curve (I. 1, Cor. 1).

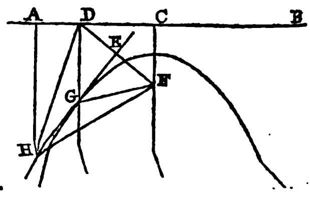
Again, the diameter GE falls wholly within the curve (figure to last proposition).

For, if any point O be assumed in it, it is evident that OD is greater than OF.

COR.—Hence, the two legs of the curve continually diverge from the axis.

PROPOSITION III.

A straight line bisecting the angle formed by two lines drawn from the same point in the curve, the one to the focus, and the other perpendicular to the directrix, is a tangent to the parabola in that point.



The straight line GE, bisecting the angle DGF, is a tangent to the parabola in G.

For, let H be any other point in GE, from which let *there be* drawn HF and HD, *also HA* perpendicular to *AB. Then, because* GE bi-

sects the vertical angle of the isosceles triangle GDF, it will also bisect the base DF at right angles (Pl. Ge. I. 4). Hence the triangles HED, HEF, are equal in every respect. Thus, HF is equal to HD, and therefore greater than HA. Consequently, the point H is without the curve (I. 1, Cor. 1).

COR. 1.—Hence the method of drawing a tangent from any point in the curve.

COR. 2.—If a straight line be drawn from the focus, to any point in the directrix, the perpendicular which bisects it will touch the parabola; also, every perpendicular to it, which cuts the curve, will be nearer to the focus than to the point in the directrix.

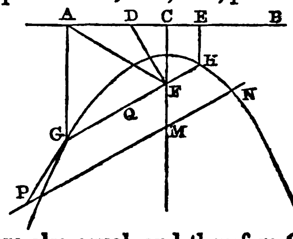
COR. 3.—The parabola is concave towards the axis.

For (Cor. 2) the tangent lies between the curve and the directrix; the curve is therefore convex towards the directrix, or concave towards the axis.

PROPOSITION IV.

Every straight line, drawn through the focus of a parabola, except the axis, meets the curve in two points.

Let FQ be a line passing through the focus, FD perpendicular to it, DA, DE, each equal to DF; AG, EH, parallel to CF, and intersected by FQ, in G, H; and let the points AF be joined. Then, because DA is equal to DF, the angles DAF, DFA, are equal; and these being taken from the right angles DAG, DFG, the remainders, the angles GAF, GFA, are equal.



Whence the sides GA, GF, are also equal, and therefore G a point in the curve (I. 1, Cor. 1). In the same manner, it may be shown that H is a point in the curve.

COR.—A straight line, making an indefinitely small angle with the axis, being produced, meets the curve; hence the rate of divergency must be very small.

For let MN be the given line. Through the focus draw GH parallel to it, then it will cut the curve in two points. *Through one of these* points, as G, draw the tangent GP.

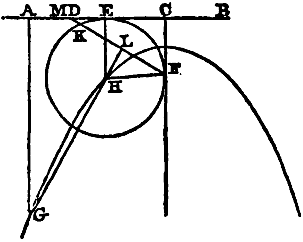
which must be inclined to GH, and therefore to MN; and will meet MN, if produced, in some point P; and therefore MN must cut the curve.

PROPOSITION V.

If, from any point in a parabola, a straight line be drawn, not parallel to a diameter, nor bisecting the angle formed by two lines drawn from that point, the one to the focus, and the other perpendicular to the directrix, it will meet the curve in one other point, and not in more than one.

Let H be a point in the curve, and HG a line not parallel to CF, nor bisecting the angle EHF; then HG shall meet the curve in another point.

For, let FL be perpendicular to GH, and it will meet AB (Pl. Ge. I. 29, Cor), since GH is not parallel to CF; also, the point D, in which it meets AB, will be different from E; since, if D, E, were the same, it would follow that HG bisects the angle EHF, contrary to the hypothesis (I. 3).



Let DA = DE, and AG parallel to EH, meet HG in G, G will be a point in the curve. For about the centre H, with the radius HE or HF, let a circle be described intersecting FD in K, and let another circle be described through the three points A, K, F. Then, because AB touches the circle EKF in E (Pl. Ge. III. 16), the rectangle $FD \cdot DK = DE^2$ (Pl. Ge. III. 36) $= DA^2$; hence DA is a tangent to the circle AKF (Pl. Ge. III. 37), and therefore AG passes through its centre; but HL, which bisects the cord FK, at right angles, also passes through its centre (Pl. Ge. III. 3); consequently G is the centre of the circle AKF. Whence GA = GF, and G a point in the parabola (I. 1, Cor. 1). If GH were to meet the curve in another point, that point would be the centre of a circle passing through F, K, and touching the line AB in a point different from A or E, which is impossible; for if it touch AB in another point, as M, then $FD \cdot DK = DM^2$;

but $FD \cdot DK = DA^2$; hence $DM^2 = DA^2$, or $DM = DA$, and M must coincide with A.

Cor. 1.—If a straight line be drawn through any point within the parabola, not parallel to a diameter, it will meet the curve in two points.

For, if the line is not parallel to the directrix, it will meet it if produced, and must therefore cut the curve in one point, as H; and hence may be found the points D, E, A, and therefore also the other point of intersection, as G.

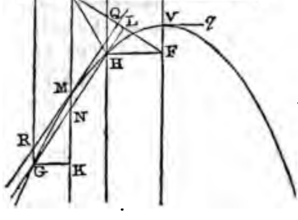
Cor. 2.—At the same point, in the curve, there cannot be more than one tangent.

Cor. 3.—A tangent bisects the angle formed by a straight line drawn to the focus, and another perpendicular to the directrix from the point of contact; it also bisects, and is perpendicular to the line that subtends that angle (Cor. 2, and I. 3).

Cor. 4.—If a straight line from the focus be perpendicular to a cord, it will bisect that part of the directrix which is intercepted by perpendiculars falling upon it from the extremities of the cord; and conversely. If FL be perpendicular to GH, it will bisect AE. For the circle EKF being described about the centre H, $FL = LK$, hence $GK = GF$, but $GF = GA$; therefore G is the centre of the circle AKF, and $DA^2 = FD \cdot DK = DE^2$.

PROPOSITION VI.

A straight line terminated by the parabola, and parallel to a tangent, is an ordinate to the diameter that passes through the point of contact.



The cord GH, parallel to the tangent MQ, will be bisected by the diameter MN.

For, let GA, HE, be perpendicular to AB, and let FD be perpendicular to and meet MQ, GH, in Q, L. Then, because FD is perpendicular to MQ, it is also perpendicular to GH, and there-

fore bisects AE in D (I. 5, Cor. 4). Whence, since AG, DN, EH, are parallel, GH is bisected in N (Pl. Ge. VI. 2).

Cor. 1.—An ordinate to any diameter is parallel to the tangent at the vertex of that diameter. For, if GN = NH, then AD = DE; therefore DF is perpendicular to GH (I. 5, Cor. 4), and GH parallel to MQ.

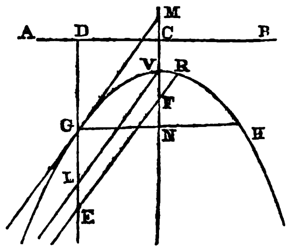
Cor. 2.—The straight line which bisects two parallel cords is a diameter.

PROPOSITION VII.

The square of a semi-ordinate, to any diameter, is equal to the rectangle under the parameter of that diameter, and the corresponding absciss.

Case 1. Let GNH be an ordinate to the axis; then $GN^2 = 4CV \cdot VN$.

For, $GN^2 = GF^2 - FN^2$: but $GF = GD = CN$; therefore $GN^2 = CN^2 - FN^2 = (CN + FN)(CN - FN)$ (Pl. Ge. II. 5, Cor.) $= 2VN \cdot CF$, and because $2VN : VN = 4CV : CF$, $2VN \cdot CF = 4CV \cdot VN$. Hence $GN^2 = 4CV \cdot VN$.



Case 2. Let GNH (figure to last proposition) be an ordinate to any other diameter MK. Then $GN^2 = 4DM \cdot MN$.

For, let the tangent MQ, the perpendiculars GA, GK, HE, and the lines DH, HF, FD, be drawn. Then, because the right-angled triangles FCD, DLN, are equiangular, and MQ, LN, parallel, $DF : FC = DN : DL = MN : LQ$, therefore $DF \cdot LQ = FC \cdot MN$ (Pl. Ge. VI. 16), and $2DF \cdot LQ$ $(= 2FC \cdot MN)$ $= 4CV \cdot MN$. But $2DF \cdot LQ = DH^2 - HF^2$ (Pl. Ge. II. c.) $= DE^2 = DA^2 = GK^2$. Consequently, $GK^2 = 4CV \cdot MN$.

Again, from the similar triangles DFC, DQM, we have $CF : FD = DQ : DM$, therefore (Pl. Ge. V. 15) $CV : DQ = DQ : DM$, and $CV : DM = DQ^2 : DM^2$ (Pl. Ge. VI. 20, Cor. 3). But, from the similar triangles GKN, DQM, $GK^2 : GN^2 = DQ^2 : DM^2$ (Pl. Ge. V. 11). Whence $GK^2 : GN^2 = CV : DM = 4CV \cdot MN : 4DM \cdot MN$. But $GK^2 = 4CV \cdot MN$, consequently $GN^2 = 4DM \cdot MN$.

COR. 1.—The square of a perpendicular, upon any diameter, from a point in the curve, is equal to the rectangle under the parameter of the axis, and the absciss corresponding to the ordinate from the same point.

For it was proved that $GK^2 = 4CV \cdot MN$.

COR. 2.—If there be two diameters, and from the vertex of each a semi-ordinate be applied to the other, the abscisses will be equal.

For, let DK, and EH, produced (figure to proposition VI.) be the diameters; then GH is an ordinate to DK, and one from M applied to EH will be the other. Let the perpendicular GK upon DK be called P; and that from M upon EH, P′; the abscissa MN, A; and that of EH, A′. Then (Cor. 1) $4A \cdot CV = P^2$, and $4A' \cdot CV = P'^2$; but since GN $=$ NH, the perpendiculars P and P′ are equal (Pl. Ge. I. 26); hence $P^2 = P'^2$; and therefore $4A \cdot CV = 4A' \cdot CV$, or $A = A'$.

COR. 3.—The square of that part of a tangent, between the point of contact and any diameter, is equal to the rectangle under the external segment of that diameter, and the parameter of the diameter which passes through the point of contact.

For $RM = GN$, and $RG = MN$ (figure to proposition VI.)

COR. 4.—The squares of ordinates, or semi-ordinates, to any diameter, are to one another as their corresponding abscisses; and the squares of perpendiculars, from the same points, are in the same ratio.

Let O, O′, be two semi-ordinates to the same diameter; A, A′, the corresponding abscisses, and P the parameter of the diameter. Then $O^2 = P \cdot A$, and $O'^2 = P \cdot A'$; therefore $O^2 : O'^2 = P \cdot A : P \cdot A' = A : A'$ (Pl. Ge. VI. 1).

COR. 5.—If the squares of parallel lines, drawn from certain points, to meet a line given in position, be to one another as the parts they cut off towards one extremity, these points will be in the curve of a parabola, which has the given line for a diameter.

For if the parallel lines be considered as ordinates, and the segments cut off from the given line as abscissæ, it follows (Cor. 4) that the given line is a diameter to a parabola passing through the extremities of the ordinates.

PROPOSITION VIII.

A subtangent, upon any diameter, is bisected in the vertex of that diameter.

Let the tangent GM meet any diameter VN in M, and let GN be an ordinate to it, from the point of contact, the subtangent MN is bisected in V (figure to proposition VII.)

For, let the diameter GL, and its semi-ordinate VL, be drawn, then is the absciss GL = VN (I. 7, Cor. 2); but, since LM is a parallelogram (I. 6, Cor. 1), GL = MV; therefore MV = VN.

Cor. 1.—That part of the axis, between the focus and any tangent, is $\frac{1}{4}$ of the parameter of the diameter passing through the point of contact.

Cor. 2.—If MV = VN, and GN a semi-ordinate, then GM is a tangent, or, if GM be a tangent, GN is a semi-ordinate.

PROPOSITION IX.

That ordinate of a diameter which passes through the focus, is equal to its parameter.

Let GE be any diameter, and RE the semi-ordinate to it, which passes through the focus; then 2RE = 4GD (figure to proposition VII).

For $RE^2 = 4GD \cdot GE$ (I. 7); but GE = FM = GD (I. 8, Cor. 1). Therefore $RE^2 = 4GD^2$, RE = 2GD, and 2RE = 4GD.

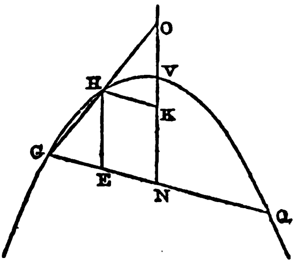
Cor.—If an ordinate to any diameter pass through the focus, the absciss will be equal to the distance of the vertex from the focus.

PROPOSITION X.

If from any point in the parabola, a parallel to a diameter be drawn to meet an ordinate to the same, the rectangle under the parameter of the diameter and the parallel will be equal to the rectangle under the segments of the ordinate.

From any point H, in the curve, let HE be drawn parallel to the diameter VN, to meet its ordinate GQ in E; then $P \cdot HE = GE \cdot EQ$.

For, let the semi-ordinate HK be drawn; then $GN^2 = P \cdot VN$ (I. 7), and $HK^2 = P \cdot VK$ (Pl. Ge. II. 5, Cor.); therefore $GN^2 - HK^2 = P \cdot KN$, or $(GN + HK) \cdot (GN - HK) = P \cdot HE$; that is, $P \cdot HE = GE \cdot EQ$.



COR. 1.—The parameter of any diameter is to the sum of two semi-ordinates as their difference to the difference of their abscisses.

For $P \cdot HE = EG \cdot EQ = (GN + HK) (GN - HK)$. Therefore (Pl. Ge. VI. 16), $P : GN + HK = GN - HK : HE$, and $HE = VN - VK$.

COR. 2.—Straight lines, drawn parallel to a diameter, from points in the curve, to meet any cord, are to one another as the rectangles under the segments of the cord.

For let P be the parameter of the diameter; L, L′, two of the parallel lines; and R, R′, the rectangles under the corresponding segments of the cord, then $P \cdot L = R$, $P \cdot L' = R'$, therefore $P \cdot L : P \cdot L' = R : R'$, or (Pl. Ge. VI. 1) $L : L' = R : R'$.

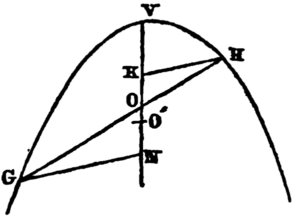
COR. 3.—If two parallel cords meet any diameter, the rectangles under their segments will be to each other directly as the parts of the diameter which they intercept from the vertex.

For Let S and S′ be the segments cut off from the diameter; R and R′ the rectangles under the segments of the cords; and P the parameter of the diameter to which the cords are ordinates. Then $P \cdot S = R$, and $P \cdot S' = R'$; therefore $R : R' = P \cdot S : P \cdot S' = S : S'$.

PROPOSITION XI.

If a diameter be cut by any straight line passing through two points in the parabola, the part intercepted from the vertex will be a mean proportional between the abscisses corresponding to the two ordinates drawn from the same points in the curve.

Let G, H, be two points in the curve, and the line GH cut the diameter VN in O; also, let the semi-ordinates GN, HK, be drawn; then VN : VO = VO : VK (see also figure to proposition X).

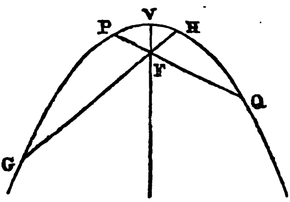


For VN : VK = GN^2 : HK^2 (I. 7, Cor. 4) = GO^2 : HO^2 = ON^2 : OK^2; and VN is divided in O and K, so that VN, VO, and VK, are in continued proportion. If not, let it be divided in O′, so that VN : VO′ = VO′ : VK; then by conversion VN : O′N = VO′ : O′K; by alternation VN : VO′ = O′N : O′K, and (Pl. Ge. VI. 22 Cor.) VN^2 : VO'^2 = $O'N^2$: $O'K^2$. But (Pl. Ge. VI. 20, Cor. 2 and 3) VN : VK = VN^2 : VO'^2, therefore VN : VK = $O'N^2$: $O'K^2$. = ON^2 : OK^2. Hence O′N : O′K = ON : OK; and, by composition, KN : O′N = KN : ON; hence O′N = ON, and the point O′ must coincide with O; therefore VN : VO = VO : VK.

PROPOSITION XII.

Any two cords which intersect each other in the focus of a parabola, are to one another directly as the rectangles under their segments.

Let the cords GH and PQ intersect one another in the focus F, then GH : PQ = GF · FH : PF · FQ.



For, since GH and PQ are equal to the parameters of the diameters to which they are ordinates (I. 9), GH · VF = GF · FH (I. 10), and PQ · VF = PF · FQ; consequently, GH : PQ = GF · FH : PF · FQ.

COR.—If two cords intersect in any point, the rectangles under their segments are to one another as the parameters of those diameters to which they are ordinates.

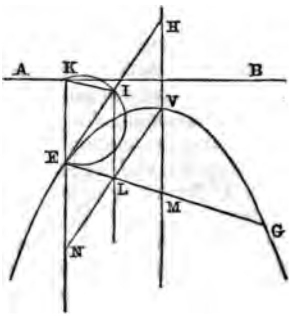
For let the rectangles under the segments of the two intersecting cords be denoted by R and R′; and the para-

meters of the diameters, to which they are ordinates, by P and P′; and L a line drawn from their point of intersection to the curve parallel to the axis; then (I. 10) $R : R' = P \cdot L : P' \cdot L = P : P'$ (Pl. Ge. VI. 1).

PROPOSITION XIII.

If from any point in a parabola, an ordinate be drawn to a diameter, and a tangent to the curve from the same point; and if a circle be described, having the ordinate for a tangent, and the fourth part of the parameter of the diameter through the given point as a cord; a diameter to the parabola passing through the point of intersection of the tangent and the circle, will cut off a fourth part of the ordinate

Let EH be a tangent at the point E, and EG an ordinate to the diameter HM of a parabola; EIK a circle, having EK for a cord, and EG for a tangent; and IL a diameter through I; then $EG = 4EL$.



The two triangles EKI, EHM, are similar, because the alternate angles KEI, EHM, are equal, and angles EKI, HEM, are equal (Pl. Ge. III. 32); therefore $KE : IE = EH : MH$, or $KE : IE = EH : 2VH$ (I. 8). Therefore $2KE \cdot VH = IE \cdot EH$, but $4KE \cdot EN = NV^2$ (I. 7) $= EH^2$; and hence $2IE \cdot EH = EH^2 = EH \cdot EH$; and therefore $2IE = EH$. Since EH is bisected in I, therefore $EL = LM$, or $4EL = EG$.

EXERCISES.

1. Any straight line, drawn from the focus of a parabola to a point in the directrix, is a mean proportional between half the parameters of the diameters which pass through its extremities.

2. If from any point in the parabola, a tangent, semi-ordinate, and perpendicular, be drawn to meet the same diameter, their squares will be to one another as the parameters of three diameters, that which passes through the point of contact, that which they meet, and the axis.

3. If through the focus of a parabola a semi-ordinate be applied to the axis, and from its extremity a tangent be drawn to meet another semi-ordinate produced; then shall the produced semi-ordinate be equal to the line joining its extremity in the curve and the focus.

4. If a tangent be drawn from any point in the curve, to meet the axis produced, and from the point of contact a perpendicular to the tangent, and a semi-ordinate to the axis, be drawn; then the segment of the axis between the perpendicular and semi-ordinate will be equal to half the parameter of the axis, and the segment between the perpendicular and tangent equal to half the parameter of the diameter which passes through the point of contact.

5. To find the directrix and focus of a parabola given in position.

6. If from the vertex of any diameter, a straight line be drawn to the extremity of a semi-ordinate so as to meet another semi-ordinate, the latter will be a mean proportional between its segments next the diameter and the former.

7. A diameter of a parabola, the tangent at the vertex of that diameter, and a point in the curve, being given, to find the directrix and focus.

8. If from any point in a tangent, a parallel to the diameter passing through the point of contact be drawn to meet an ordinate of the same, the rectangle under the parallel, and its external segment, will be equal to the square of that part of it that is between the curve, and the line joining the vertex of the diameter, and either extremity of its ordinate.

9. The focus and directrix of a parabola being given, to draw a tangent to the curve parallel to a line given in position that is not perpendicular to the directrix.

10. If there be two tangents to a parabola, such, that the straight line joining their points of contact pass through the focus, they will cut each other at right angles, their intersection will be in the directrix, and the line joining it with the focus will be perpendicular to the line joining the points of contact.

11. From a given point, in a given parabola, to draw a tangent without finding the focus.

12. Two parabolas of equal parameters, having their axes in the same line, and in the same direction, but their ver-

tices at different points, being produced indefinitely, continually approach, but never meet.

13. In a given parabola to find a diameter that makes a given angle with its ordinates.

14. If from any point in a tangent, a straight line be drawn, to meet the curve and the diameter passing through the point of contact, the square of its segment, between the tangent and the diameter, will be equal to the rectangle under its segments between the tangent and the points in the curve.

15. The focus and directrix of a parabola being given, to draw a tangent to the curve from a given point without it.

16. If from a point in a parabola, a semi-ordinate be applied to a diameter, and from the same point any other straight line be drawn to meet it, the square of this line will be equal to the rectangle under the absciss of that diameter, and the parameter of the diameter to which the straight line, when produced, is an ordinate.

17. If from points in the curve, two tangents be drawn to meet, their squares will be to each other as the parameters of the diameters passing through the points of contact.

18. If two tangents, and the line joining their points of contact, meet the same diameter, the segment of the diameter, intercepted by this line from the vertex, will be a mean proportional between those intercepted by the tangents.

SECOND BOOK.

ELLIPSE.

DEFINITIONS.

1. If two points be given in position, the locus of a point, the sum of whose distances from them is always the same, is a curve called an *ellipse*.

2. The given points are named the *foci*, and the middle of the line that joins them, the *centre* of the ellipse.

3. The distance of the centre from one of the foci is called the *eccentricity*.

4. A *diameter* is a straight line drawn through the centre, and terminated on both sides by the curve.

5. The diameter which passes through the foci is named the *transverse* or *major axis*, and that which is perpendicular to it, the *conjugate* or *minor axis*.

6. An *ordinate* to a diameter is a straight line not passing through the centre, but terminated by the curve, and bisected by the diameter.

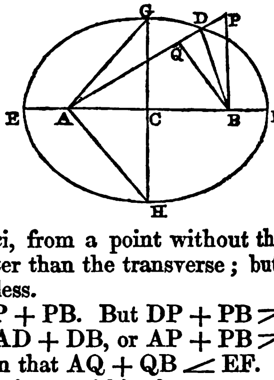
7. Two diameters are said to be *conjugate* to one another when each is parallel to the ordinates of the other.

8. The *parameter* of a diameter is a third proportional to that diameter and its conjugate.

PROPOSITION I.

If from any point in an ellipse, straight lines be drawn to the foci, their sum is equal to the transverse axis.

Let EFGH be an ellipse, of which EF is the transverse, and GH the conjugate axis, A, B, the foci, and D any point in the curve; then AD + DB = EF.



For EA + EB = FA + FB; hence AE = BF, and AD + DB = EA + EB (II. Def. 1) = EF.

Cor. 1.—If two straight lines be drawn to the foci, from a point without the ellipse, their sum is greater than the transverse; but, if from a point within it, less.

For AP + PB = AD + DP + PB. But DP + PB $>$ DB; therefore AP + PB $>$ AD + DB, or AP + PB $>$ EF. And it is similarly shown that AQ + QB $<$ EF.

Cor. 2.—A point is without, in, or within the curve, according as the sum of the lines drawn to the foci is greater, equal, or less, than the transverse axis.

Cor. 3.—The distance of either extremity of the conjugate axis, from one of the foci, is equal to half the transverse.

For if GB be joined, the triangles ACG, BCG, are equal (*II. Def.* 2 and 5, and Pl. Ge. I. 4); therefore AG = GB,

But EF = AG + GB = 2AG or 2GB. It is similarly proved that EF = 2AH or 2BH.

COR. 4.—The transverse and conjugate axis are bisected in the centre.

AC = CB (II. Def. 2), and EA = BF; therefore EC = CF. Also in the triangles ACG, ACH, AG = AH, therefore the angles at G and H are equal, those at C are right angles, and AC is common; hence (Pl. Ge. I. 26) CG = CH.

COR. 5.—A perpendicular to the transverse at one of its extremities is a tangent to the ellipse.

For if I be a point in that perpendicular, AI $>$ AF, and BI $>$ BF; therefore AI + IB $>$ AF + FB or EF.

COR. 6.—The square of half the conjugate axis is equal to the rectangle under the segments into which the transverse is divided in one of the foci.

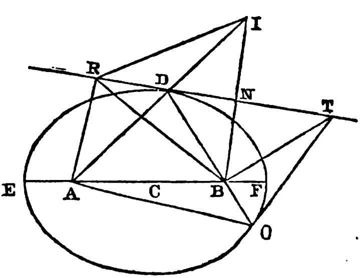
For $CG^2 = AG^2 - AC^2 = EC^2 - AC^2 = EA \cdot AF$, or $EB \cdot BF$ (Pl. Ge. II. 5, Cor.)

COR. 7.—The distance of the foci is a mean proportional between the sum and difference of the transverse and conjugate axis.

For $AC^2 = AG^2 - GC^2 = (AG + GC)(AG - GC)$ (Pl. Ge. II. 5, Cor), or $AG + GC : AC = AC : AG - GC$; and the doubles of these terms are also proportional.

PROPOSITION II.

The straight line which bisects the angle adjacent to that which is contained by two straight lines, drawn from any point in the ellipse to the foci, is a tangent to the curve in that point.



Let D be any point in the curve, from which AD, DB, are drawn to the foci, and let the angle BDI adjacent to ADB, be bisected by the line DT; then is DT a tangent to the ellipse in the point D.

For, let any other

point R be assumed in DT, and DI being made equal to DB, let BI, RA, RB, and RI, be drawn. The line DNT, which bisects the vertical angle of the isosceles triangle BDI, also bisects the base BI at right angles. Hence $RB = RI$ (Pl. Ge. I. 4), and $AR + RB = AR + RI$, and therefore greater than AI (Pl. Ge. I. 20) or EF (II. 1). Consequently the point R is without the ellipse (II. 1, Cor. 2).

Cor. 1.—A perpendicular to the conjugate axis, at one of its extremities, is a tangent to the ellipse.

For this line bisects the angle adjacent to that formed by lines drawn to the foci from this point.

Cor. 2.—The method of drawing a tangent from a given point in the curve, also of drawing a tangent parallel to a line given in position, is evident.

When the tangent is to be parallel to a given line, draw from the focus B a line BI perpendicular to the line, and make $AI = EF$; join AI, and D will be the point of contact.

Cor. 3.—There cannot be more than one tangent to the ellipse at the same point.

For the sum $AD + DB$ of the lines from A and B, to a point D in RN, which make equal angles with it, is less than the sum of any other two lines drawn from A and B to any other point as R in RN. Now, if another tangent can be drawn through D, AD and DB would not make equal angles with it, but some other two lines AD′, D′B, to some point D′ in it, would make equal angles with it. But $AD' + D'B$ would be less than $AD + DB$, and hence the point D′ would be within the curve (II. 1, Cor. 2), and the line would not be a tangent.

Cor. 4.—Every tangent bisects the angle adjacent to that contained by straight lines drawn to the foci from the point of contact, or, which is the same, these lines make equal angles with the tangent.

Cor. 5.—A straight line drawn from the centre to meet a tangent, and parallel to the line joining the point of contact, and one of the foci, is equal to half the transverse axis.

For, since $BC = CA$, and $BN = NI$, the line that joins CN will be parallel to AD, and equal to the half of AI.

Cor. 6.—A perpendicular to a tangent, from one of the

foci, and a parallel to the line joining the other focus, and the point of contact from the centre, meet the tangent in the same point.

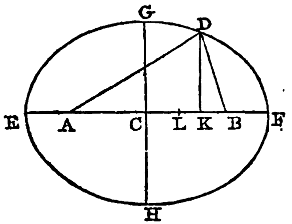
COR. 7.—If a cord pass through one of the foci, and the tangents at its extremities be produced to meet, the straight line that joins the point of concourse and the focus, will be perpendicular to the cord.

Let the tangents at the extremes of the cord DBO, passing through the focus, meet in the point T, and let a line TI be perpendicular to AD; then, because DT, TO, bisect the exterior angles of the triangle ADO, AI is equal to half its perimeter (Pl. Ge. VI. H) = AD + DB. Hence DB = DI, and triangles TDI, TDB, equal in every respect, and angle TBD = TID, or a right angle.

PROPOSITION III.

From any point in an ellipse, a perpendicular being let fall upon the transverse, and straight lines drawn to the foci, as half the transverse is to the eccentricity, so is the distance of the centre from the perpendicular to half the difference of the straight lines drawn to the foci.

Let D be the point in the curve, from which DA, DB, are drawn to the foci, and DK perpendicular to EF, then CF : CB = CK : $\frac{1}{2}$ (AD — DB).



For, let EL = AD, then LF = DB, and 2CL = AD — DB. But (AD + DB) · (AD — DB) = AB · 2CK (Pl. Ge. II. c), or EF · 2CL = AB · 2CK; therefore EF : AB = 2CK : 2CL, and CF : CB = CK : CL.

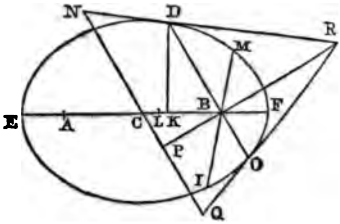
PROPOSITION IV.

Any two cords which pass through one of the foci of an ellipse, are to one another directly as the rectangles under their segments.

Let DO, MI, be the two cords passing through the focus B, DO : MI = DB · BO : MB · BI.

For, let the tangents at D, O, meet each other in R,

through C let NQ be drawn parallel to DO, meeting the tangents in N, Q, and let RB meet NQ in P. It has been proved that CN or CQ $=$ CF (II. 2, Cor. 5); therefore NQ $=$ EF; that CF : CB $=$ CK : CL (II. 3), therefore CF · CL $=$ CB · CK, and that DBR, or DBP, is a right angle (II. 2, Cor. 7); therefore the triangles DBK, BCP, are similar, and DB · CP $=$ BC · BK (Pl. Ge. VI. 4 and 16). Hence $CF^2 = CF \cdot CL + CF \cdot LF$ (Pl. Ge. II. 2) $= BC \cdot CK + DB \cdot CQ = BC \cdot CK + BC \cdot BK + BD \cdot PQ = BC^2 + DB \cdot PQ$, and $DB \cdot PQ = CF^2 - BC^2 = EB \cdot BF$ (Pl. Ge. II. 5, Cor).



But $DO : NQ = BO : PQ = DB \cdot BO : DB \cdot PQ$.
Therefore $DO : EF = DB \cdot BO : EB \cdot BF$.
In like manner $EF : MI = EB \cdot BF : MI \cdot BI$.
Consequently $DO : MI = DB \cdot BO : MB \cdot BI$ (Pl. Ge. V. 24).

COR.—The rectangle under any cord passing through the focus and the parameter of the transverse, is equal to four times the rectangle under its segments.

For, since $EF : DO = EB \cdot BF : DB \cdot BO$, if P be the parameter of the transverse, $EF \cdot P : DO \cdot P = 4EB \cdot BF : 4DB \cdot BO$. But $EF \cdot P =$ * GH^2 (II. Def. 8) $= 4EB \cdot BF$ (II. 1, Cor. 6); therefore $DO \cdot P = 4DB \cdot BO$ (Pl. Ge. V. 14).

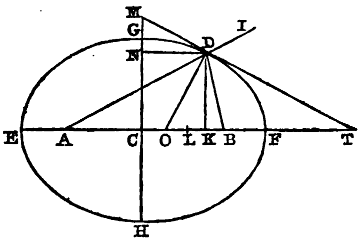
PROPOSITION V.

If a tangent to the ellipse meet either axis produced, and a perpendicular to the axis be drawn from the point of contact, then shall the semi-axis be a mean proportional between its segments intercepted from the centre by the tangent and perpendicular.

Case 1. Let the tangent DT, and the perpendicular DK, meet the transverse in T, K; then $CT : CF = CF : CK$.

* *That EF, GH, are conjugate diameters, will be proved independent of this Cor. For GH, see last figure.*

For, since DT bisects the angle BDI, AT : TB = AD : DB



(Pl. Ge. VI. A), and $\frac{1}{2}$ (AT + TB) : $\frac{1}{2}$ (AT — TB) = $\frac{1}{2}$ (AD + DB) : $\frac{1}{2}$ (AD — DB) (Pl. Ge. V. 15, and G).

That is CT : CB = CF : CL.

But CB : CF = CL : CK (II. 3).

Therefore CT : CF = CF : CK (Pl. Ge. V. 24).

Case 2. Let the tangent DT, and the perpendicular DN, meet the conjugate in M, N; then MC : CG = CG : CN.

For, let DO, perpendicular to DT, meet the transverse in O, then DO evidently bisects the angle ADB; therefore AO : OB = AD : DB (Pl. Ge. VI. A), hence AT : TB = AO : OB, and CT : CB = CB : CO (Pl. Ge. V. 15, and G); therefore CT · CO = CB^2, but CT · CK = CF^2, consequently CT · KO = CF^2 — CB^2 = CG^2.

Again, from the similar triangles CTM, KDO, CT · KO = CM · DK (Pl. Ge. VI. 4 and 16) = CM · CN; therefore CM · CN = CG^2, and CM : CG = CG : CN.

COR. 1.—The rectangles under the segments of the axis, and the segments of the semi-axis produced to meet the tangent, both being divided by the perpendicular, are equal.

For EK · KF = CF^2 — CK^2 = TC · CK — CK^2 = CK · KT. And similarly HN · NG = CN · NM.

COR. 2.—The segments into which the axis is divided by the tangent, are directly proportional to the segments into which it is divided by the perpendicular.

For, since CT : CF = CF : CK, CT + CF : CT — CF = CF + CK : CF — CK, that is, ET : TF = EK : KF.

Cor. 3.—The segments CT, CB, CO, are proportional

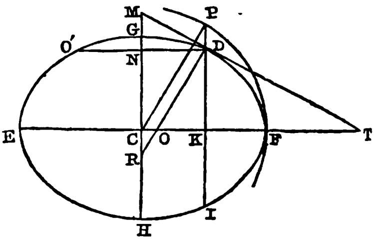
(proved in case 2); and, in like manner, it may be proved that CK, CL, CO, are proportional.

PROPOSITION VI.

If from any point in an ellipse, a perpendicular fall upon either axis, as the square of that axis is to the square of the other, so is the rectangle under the segments of the former, to the square of the perpendicular.

Let the perpendicular DK fall from any point in the curve upon EF, then $EF^2 : GH^2 = EK \cdot KF : DK^2$.

For, since $CT : CF = CF : CK$ (II. 5), $CF^2 : CK^2 = CT : CK$ (Pl. Ge. VI. 20, Cor. 2 and 3), and, by conversion, $CF^2 : EK \cdot KF = CT : TK$. But $CT : TK = CM : DK$ or $CN = CG^2 : CN^2$, or DK^2; therefore $CF^2 : EK \cdot KF = CG^2 : DK^2$, and alternately $CF^2 : CG^2 = EK \cdot KF : DK^2$. Consequently $EF^2 : GH^2 = EK \cdot KF : DK^2$. If the perpendicular DN fall upon GH, it may be proved in the same manner that $GH^2 : EF^2 = GN \cdot NH : DN^2$.



COR. 1.—The squares of perpendiculars, falling from points in the curve, upon either axis, are to one another as the rectangles under the segments into which they divide the axis.

For the squares of the perpendiculars are to the rectangles in the same ratio as the squares of the axes; and therefore the squares of the perpendiculars are proportional to these rectangles.

COR. 2.—If a circle be described upon the transverse, and another upon the conjugate axis, the former will contain, and the latter will be contained in, the ellipse, so that the curve lines shall touch one another only in the extremities of their common diameter.

For $DK^2 : EK \cdot KF = GH^2 : EF^2$; but $GH \angle EF$; *therefore* $DK^2 \angle EK \cdot KF$. Now, if PF be an arc of the

circle described on EF, $PK^2 = EK \cdot KF$; therefore $DK \angle PK$, and the point P is without the ellipse, and the circle meets the ellipse only at E and F. It may be similarly shown that the circle described on the conjugate diameter is within the ellipse.

Cor. 3.—*The two axes are the greatest and least diameters of the ellipse.*

For, any diameter of the ellipse, except EF, is less than the diameter of the circle described on EF, and hence EF is the greatest diameter. So any diameter of the ellipse except GH, is greater than the diameter of the circle described on GH; hence GH is the least diameter of the ellipse.

Cor. 4.—*If a circle be described upon either axis of an ellipse, perpendiculars to the common diameter are cut proportionally by the curves.*

For $DK^2 : EK \cdot KF = GH^2 : EF^2$; but $PK^2 = EK \cdot KF$; therefore $DK^2 : PK^2 = GH^2 : EF^2$, or $DK : PK = GH : EF$, or the perpendiculars in the ellipse are to the corresponding ones in the circle, in the ratio of the axes, that is, in a constant ratio.

Cor. 5.—*Every straight line terminating in the curve, and parallel to one axis, is an ordinate to the other; and conversely.*

For it was shown that $EF^2 : GH^2 = EK \cdot KF : DK^2$; and it may similarly be proved that $EF^2 : GH^2 = EK \cdot KF : KI^2$; and therefore $DK = KI$, and DI is an ordinate to EF. It may be similarly shown that DO′ is an ordinate to GH.

Cor. 6.—*The two axes are conjugate diameters.*

Cor. 7.—*The ordinate through the focus is the parameter of the transverse.*

For, if K be the focus, $EK \cdot KF = GC^2$, but $EC^2 : GC^2 = EK \cdot KF : DK^2 = GC^2 : DK^2$ (II. Def. 8).

Cor. 8.—*Equal ordinates to either axis are equally distant from the centre; the greater ordinate is nearer the centre; and conversely.*

For $EK \cdot KF = CF^2 - CK^2$; and therefore the less CK is, the greater is $EK \cdot KF$, and hence the greater is DK.

Cor. 9.—*If from any point in the curve, a straight line be drawn to the conjugate axis, equal to half the*

transverse, its segment intercepted from the same point by the transverse, will be equal to half the conjugate; and conversely.

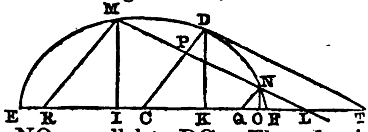
For RD = CP = CF, and RD : OD = CP : OD = PK : DK = CF : CG, or RD : OD = CF : CG, but RD = CF, therefore OD = CG.

Schol.—On this principle, the elliptic compasses are constructed. If EF and GH represent two bars of brass with grooves, and RD a third bar equal to CF, and the part of it OD equal to CG, then two projecting pins at R and O rest in the grooves; and if RD be moved, so that the pin at R will move in the groove of GH, and that at O in the groove of EF, the extremity D will describe an ellipse.

PROPOSITION VII.

A straight line, not passing through the centre, but terminated by the ellipse, and parallel to a tangent, is an ordinate to the diameter that passes through the point of contact.

The cord MN, parallel to the tangent at D, is an ordinate to the diameter CD.



For, let MN meet CT in L, and from the points M, N, let MI, NO, be drawn perpendicular to EF and MR, NQ, parallel to DC. Then, by similar triangles, MI : DK = RI : CK.

And MI : DK = IL : KT; therefore (Pl. Ge. VI. 23, Cor.)

$MI^2 : DK^2 = RI \cdot IL : CK \cdot KT$.

Hence $EI \cdot IF : EK \cdot KF$ (II. 6, Cor. 1) $= RI \cdot IL : CK \cdot KT$.

But $EK \cdot KF = CK \cdot KT$ (II. 5, Cor. 1).

Therefore $EI \cdot IF = RI \cdot IL$.

And $EI : IL = RI : IF$.

By composition $EL : IL = RF : IF$.

By alternation and conversion $EL : ER + FL = IL : FL$.

In like manner $EL : EQ + FL = OL : FL$.

Hence by equality $IL : OL = EQ + FL : ER + FL$.

And $RL : QL = EQ + FL : ER + FL$.

By division $RQ : QL = RQ : ER + FL$.

Consequently QL = ER + FL, ER = FQ, CR = CQ, and MP = PN.

Cor. 1.—Every ordinate to a diameter is parallel to the tangent at its vertex.

For, if not, let a tangent be drawn parallel to it, then the diameter through the point of contact would bisect it; and thus the same line would be bisected in two different points, which is absurd.

Cor. 2.—All the ordinates to the same diameter are parallel to one another.

Cor. 3.—A straight line, drawn through the vertex of a diameter, parallel to its ordinate, is a tangent to the curve in that point.

Cor. 4.—The diameter which bisects one of two parallel chords, will also bisect the other.

Cor. 5.—The straight line which bisects two parallel chords, and terminates in the curve, is a diameter.

Cor. 6.—If two tangents be at the vertices of the same diameter, they are parallel; and conversely.

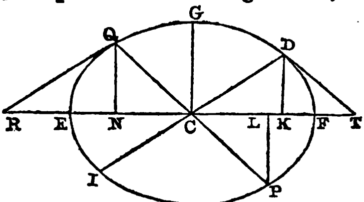
Cor. 7.—Every diameter divides the ellipse into two equal parts.

PROPOSITION VIII.

Two diameters, one of which is parallel to the tangent, at the vertex of the other, are conjugate to one another.

Let the diameter PQ be parallel to the tangent DT, at the vertex of the diameter DI; then shall PQ, DI, be conjugate diameters.

For, let DK, QN, be perpendicular to EF, and let the tangents at D, Q, meet EF in T, R.



The triangles TDK, CQN, by reason of parallel lines, are equiangular, therefore TK : CN = DK : QN; hence TK^2 : $CN^2 = DK^2 : QN^2$ = EK · KF : EN · NF (II. 6, Cor. 1) = *CK · KT : RN* · NC (II. 5, Cor. 1), and TK : CN =

$CK : RN$, or $DK : QN = CK : RN$. Consequently the triangles DKC, RNQ, are equiangular, and DC parallel to RQ. Whence PQ, DI, are each parallel to the ordinates of the other (II. 7, Cor. 1), that is, they are conjugate diameters.

COR. 1.—If from the extremities of two conjugate semi-diameters, perpendiculars be let fall upon either axis, the rectangle under the segments, into which the axis is divided by one of the perpendiculars, is equal to the square of the segment, which the other intercepts from the centre.

For, since RQ is parallel to CD, $CT : TK = CR : CN$, and $CT \cdot CK : CK \cdot KT = CR \cdot CN : CN^2$. But $CT \cdot CK = CF^2 = CR \cdot CN$, therefore $CK \cdot KT$, or $EK \cdot KF = CN^2$.

By letting fall perpendiculars upon the conjugate axis from the extremities of the same conjugate diameters, the same property may be similarly proved in reference to it.

COR. 2.—Also, the sum of the squares of the segments, intercepted from the centre, is equal to the square of half the axis upon which the perpendiculars fall; and the sum of the squares of the perpendiculars is equal to the square of half the other axis.

$CN^2 = EK \cdot KF$ (Cor. 1) $= CF^2 - CK^2$ (Pl. Ge. II. 5, Cor.); therefore $CN^2 + CK^2 = CF^2$.

It may be similarly proved that $QN^2 + DK^2 = CG^2$; for QN, DK, are equal to the segments intercepted on CG from C by perpendiculars on it from Q and D.

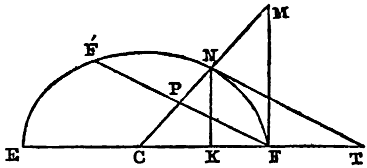
COR. 3.—The sum of the squares of any two conjugate diameters is equal to the sum of the squares of the transverse and conjugate axis.

For $CD^2 + CQ^2 = CK^2 + KD^2 + CN^2 + NQ^2 = (CK^2 + CN^2) + (KD^2 + NQ^2) = CF^2 + CG^2$.

PROPOSITION IX.

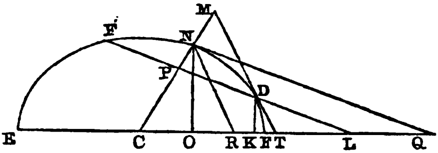
Any semi-diameter which meets an ordinate and tangent drawn from the same point in the ellipse, is a mean proportional between the segments intercepted from the centre.

Case 1. Let the semi-diameter CN meet a tangent and ordinate drawn from F, one extremity of either axis, in M, P. Then CM : CN = CN : CP.



For, let the tangent and ordinate from N meet the axis EF in T, K. It has already been demonstrated, with respect to either axis, that CT : CF = CF : CK; hence CM : CN = CF : CK = CT : CF = CN : CP.

Case 2. Let the semi-diameter CN meet a tangent and ordinate from any other point D in M, P, CM : CN = CN : CP.



For, let DP, the tangent at N, DM, and NR parallel to DM, meet the axis EF in L, Q, T, R. Also, let DK, NO, be the semi-ordinates to EF from D, N. The lines DK, DT, DL, are respectively parallel to NO, NR, NQ.

Hence $CO \cdot OQ : CK \cdot KT =$ (II. 5, Cor. 1, and 6, Cor. 1) $NO^2 : DK^2 = OQ^2 : KL^2$.

By alternation $CO : OQ = CK \cdot KT : KL^2$.

But $OQ : OR = KL : KT = KL^2 : KL \cdot KT$.

Therefore $CO : OR = CK : KL$ (by equality).

By composition $CO : CR = CK : CL$.

Again (II. 5) $CT \cdot CK = CF^2 = CQ \cdot CO$.

Therefore $CT : CO = CQ : CK$.

Hence $CT : CR = CQ : CL$ (by equality).

And therefore $CM : CN = CN : CP$.

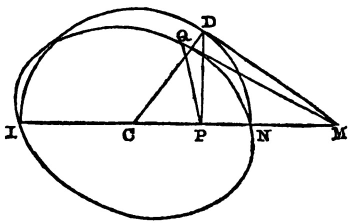
COR. 1.—*Tangents, at the extremities of an ordinate, meet its diameter produced in the same point.*

For if a tangent from F′ meet CN produced in a point M′, it may similarly be shown that CP : CN = CN : CM′; therefore CM′ = CM, or the tangents from F and F′ meet in the same point M.

COR. 2.—*A straight line, drawn through the centre, from the point of intersection of two tangents, bisects the line that joins the points of contact.*

This is the converse of the last corollary.

COR. 3.—*If a circle be described upon any diameter of an ellipse, and through any point of it an ordinate of each curve be drawn, tangents, at their extremes, will meet one another in the common diameter produced.*



For in the ellipse CP : CN = CN : CM. And if the tangent to the circle at D meet CN produced in some point M′, then, from similar triangles, CP : CD = CD : CM′, or CP : CN = CN : CM′; therefore CM′ = CM, or the tangents meet in the same point.

COR. 4.—*The segments into which any diameter is divided by a tangent, are directly proportional to the segments into which it is divided by an ordinate from the point of contact.*

For CM : CN = CN : CP, and by mixing IM : MN = IP : PN; or the diameter is cut internally and externally in the same ratio; or the diameter produced is cut harmonically.

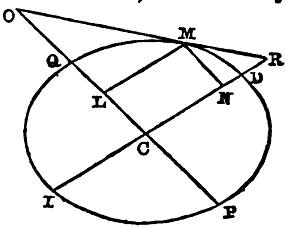
COR. 5.—*If a semi-diameter be produced to meet a tangent, the rectangle under the segments into which it is divided, by an ordinate from the point of contact, is equal to the rectangle under the segments, into which the whole diameter is divided by the same ordinate.*

For $CN^2 = CP \cdot CM = CP^2 + CP \cdot PM$; therefore $CP \cdot PM = CN^2 - CP^2 = IP \cdot PN$.

PROPOSITION X.

mi-ordinate be applied to any diameter of an ellipse, re of that diameter is to the square of its conjugate, ctangle under its segments to the square of the inate.

Q, DI, be two conjugate diameters, and MN any inate to DI; DI^2 : O
)N · NI : MN^2.



et the tangent at M , PQ in R, O, and let , semi-ordinate to PQ.)ecause CR : CD = (II. 9), $CD^2 : CN^2$ = , and CD^2 : DN · NI = (Pl. Ge. V. D, and II. = OC : CL = $CQ^2 : CL^2$. Hence $CD^2 : CQ^2$ = : MN^2, and $DI^2 : PQ^2$ = DN · NI : MN^2.

l.—The squares of semi-ordinates are to one another he rectangles under the segments of the diameter.

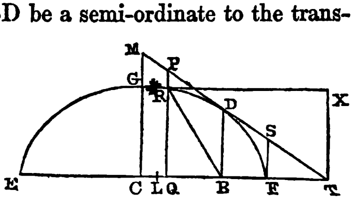
2.—Any diameter of an ellipse is to its parameter he rectangle under its segments to the square of semi-ordinate that divides them.

et P be the parameter of the diameter DI, then , = PQ : P, and (Pl. Ge. VI. 20, Cor. 2 and 3) = $DI^2 : PQ^2$ = IN · ND : MN^2.

PROPOSITION XI.

)ugh any point in the ellipse, a parallel to the con- xis meet the transverse, and the focal tangent, its between these will be equal to the distance of that m the focus.

: focus B, let BD be a semi-ordinate to the trans- ıen DT, touch- ellipse in D, is ıe focal tangent. any point R in :, let RQ, paral- !, meet DT, EF, ,; PQ is equal



For, let $FL = RB$.

Then, because $CT : CF = CF : CB$ (II. 5) $= CQ : CL$ (II. 3).

Alternately $CT : CQ = CF : CL$.

By conversion $CT : TQ = CF : FL$ or RB.

Hence $MC : PQ = CF : RB$.

But $MC = CF$ (II. 2, Cor. 5), consequently $PQ = RB$.

Cor. 1.—A perpendicular to the transverse, at either extremity, limited by the focal tangent, is equal to the distance of that extremity from the focus.

For (II. 5) $CT : CF = CF : CB$; and, by conversion, $CT : TF = CF : FB$; but $CT : TF = CM : FS$, and $CM = CF$ (II. 2, Cor. 5), therefore $FB = FS$.

Cor. 2.—The distance of any point in the curve, from the perpendicular to the transverse, at the point where it intersects the focal tangent, is to its distance from the focus as half the transverse is to the eccentricity.

For $TQ : PQ = CT : CM = CT : CF = CF : CB$ (II. 5) or $RX : RB = CF : CB$.

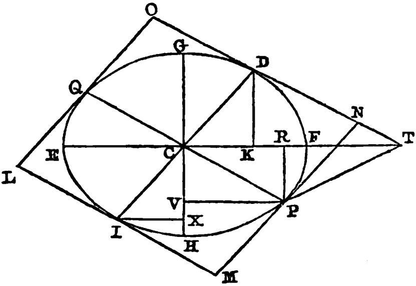
Schol.—The property of the ellipse stated in this corollary, is, in some treatises on the conic sections, assumed as the characteristic property of the curve, on which to found its definition. The perpendicular XT is then called the *directrix*, and the ratio of CB to CF the *determining ratio*. The ellipse may then be defined to be a curve, such that the distance of any point in it from the focus, is to its distance from the directrix in a constant ratio, that is, in the determining ratio, or the ratio of the *eccentricity* to the *semi-transverse*.

PROPOSITION XII.

If four straight lines touch an ellipse at the vertices of two conjugate diameters, the parallelogram which they contain will be equal to the rectangle under the transverse and conjugate axis.

Let DI, PQ, be two conjugate diameters, and LM, MN,

NO, OL, tangents at their vertices. Then LMNO is a parallelogram (II. 7, Cor. 6), divided into four equal parallelograms by PQ, DI; and the whole MO is equal to the rectangle under EF and GH, or the part $CN = CF \cdot CG$.



For, let DK, PR, be perpendicular to EF, and let EF meet the tangent DN in T, and TP be joined.

Then, since (II. 6) $CF^2 : CG^2 = (ER \cdot RF =) \ CK^2 : PR^2$ (II. 8, Cor.), hence $CF : CG = CK : PR$ (II. 9, Cor. 5).

But $\quad CT : CF = CF : CK$ (II. 5)

Therefore $\quad CT : CG = CF : PR$ (Pl. Ge. V. 22).

Hence $CF \cdot CG = CT \cdot PR =$ double the triangle CTP = the parallelogram CN.

Cor. 1.—If perpendiculars fall from the extremities of two conjugate semi-diameters upon either axis, the axis upon which they fall is to the other as the segment intercepted by one of the perpendiculars from the centre is to the other perpendicular.

For it was proved that $CF : CG = CK : PR$; therefore, since $PR = CV$, and $CK = IX$, $CG : CF = CV : IX$.

Cor. 2.—All parallelograms whose sides touch an ellipse in the vertices of two conjugate diameters, are of the same magnitude.

Cor. 3.—Every parallelogram whose angular points are the vertices of two conjugate diameters, is equal to half the rectangle under the axes.

For if D, P, I, and Q, were joined, the parallelogram so formed would be the half of OM.

EXERCISES.

1. The rectangle under perpendiculars to the transverse passing through its extremities, the rectangle under perpen-

diculars from the foci to any tangent, the rectangle under perpendiculars to the transverse, from the centre and the point of contact, and the rectangle under perpendiculars to the tangent, from the point of contact, and the centre (all the perpendiculars being limited by the transverse and tangent), are each equal to the square of the semi-conjugate axis.

2. If from any point in an ellipse, two straight lines be drawn through the extremities of a diameter, any other straight line parallel to the tangent at that point, and terminating in these lines, will be bisected by an ordinate to that diameter, from the point of contact.

3. If the diagonals of a quadrilateral figure, inscribed in an ellipse, intersect one another in one of the foci, and two opposite sides be produced to meet, the straight line that joins the point of concourse and the foci will bisect the angle formed by these diagonals.

4. If two tangents, at the vertices of a diameter, meet any third tangent, the rectangle under the two former, and the rectangle under the segments of the latter, from the point of contact, are respectively equal to the squares of the semi-diameters to which the tangents are parallel.

5. If through the extremities of the transverse axis, two tangents be drawn to meet a third, the circle described upon the intercepted tangent will pass through the foci.

6. If from the vertices of two conjugate semi-diameters, two semi-ordinates be applied to any diameter, the square of its segment, between the centre and either semi-ordinate, will be equal to the rectangle under the segments into which it is divided by the other semi-ordinate.

7. If from the vertex of any diameter, straight lines be drawn to the foci, their rectangle is equal to the square of its semi-conjugate.

8. If a tangent meet two conjugate diameters, the rectangle under its segments, from the point of contact, will be equal to the square of the semi-diameter to which it is parallel.

9. Every diameter is a mean proportional between the ordinate to its conjugate, which passes through the focus and the transverse axis.

10. If from any point in one tangent, a straight line be drawn parallel to another, to meet the curve in two points, the rectangle under its segments, between the tangent and the curve, is equal to the square of its segment between the tangent and the line joining the points of contact.

11. The rectangles under the segments of two cords which intersect, are to one another directly as the squares of the diameters to which they are parallel.

12. If from any point in a given ellipse, straight lines, parallel to lines given in position, be drawn to meet the sides of a given quadrilateral inscribed in it, the rectangles under those drawn to opposite sides, will have to each other a given ratio.

13. The distance between the directrix (Schol. to Prop. 11), and the focus next it, is divided by the curve internally and externally in the same ratio; or the distance between the directrix, and the farther extremity of the transverse, is divided harmonically by the other extremity of this diameter and the focus next it.

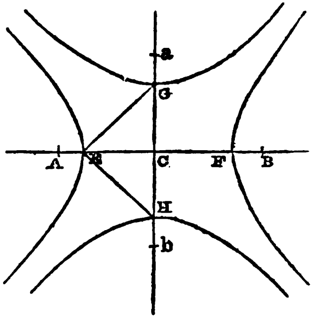
14. If from one focus as a centre with the transverse as a radius, a circle be described, every point in the ellipse will be equally distant from this circle and the other focus.

THIRD BOOK.

HYPERBOLA.

DEFINITIONS.

1. If two points be given in position, the locus of a point, the difference of whose distances from them is always the same, is a curve called a *hyperbola*.



Thus A and B being the given points, the curve passing through F is a hyperbola. If another similar curve pass through E, so that AE = BF, these two branches are called *opposite* hyperbolas.

2. The given points are named the *foci;* that part of the line that joins them, intercepted by opposite hyperbolas, the *transverse* axis; and the middle of the same line, the *centre.*

3. The *conjugate* axis is a straight line, passing through the centre, perpendicular to the transverse, and limited by a circle described from one extremity of the transverse with the distance of either focus from the centre as a radius.

4. If two other hyperbolas be described, having the conjugate for their transverse axis, and their foci at the same distance from the centre as the foci of the former, these are also called *opposite hyperbolas,* and have the transverse of the former for their conjugate axis. These are the branches through G and H.

5. A *diameter* is a straight line drawn through the centre, and terminated by opposite hyperbolas.

6. An *ordinate* to a diameter is a straight line, terminated by the hyperbola, and bisected by that diameter produced.

7. An *external ordinate* to a diameter is a straight line, terminated by opposite hyperbolas, and bisected by that diameter, or the same produced.

8. Two diameters are said to be *conjugate* to one another when they are mutually parallel to each other's ordinates.

9. A third proportional, to any diameter, and its conjugate, is called its *parameter.*

10. The *eccentricity* is the distance between the centre and either focus.

11. An *assymptote* of the hyperbola is a straight line, *which, being* produced indefinitely, does not meet, but con-

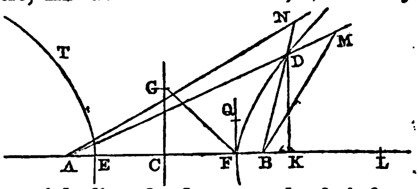
tinually approaches the curve, so as to come within less than any given distance from it.

12. When the assymptotes are perpendicular, the hyperbola is said to be *rectangular*.

PROPOSITION I.

If from a point in any hyperbola, straight lines be drawn to the foci, their difference is equal to the transverse axis.

Let DF, ET, be opposite hyperbolas, of which A, B, are the foci, C the centre, EF the transverse axis, and D any point in one of the curves; then $AD - DB = EF$.



For $AD - DB = AF - FB$ (III. Def. 1) $= AE + EF - FB = EF$.

Cor. 1.—If two straight lines be drawn to the foci, from a point without the opposite hyperbolas, their difference is less than the transverse axis, but, if from any point within either of them, it is greater.

Let M be a point within the hyperbola DF, and N a point without, between the curve and its conjugate axis. The lines AM, NB, necessarily meet the curve; let them meet it in the points D, D; then, since $MD + DB > MB$, $AM - MB$ is greater than $AM - (MD + DB)$, or EF, but since $AD + DN > AN$, $AN - NB$ is less than $(AD + DN) - NB$, or EF.

Cor. 2.—A point is within, in, or without, the curve, according as the difference of its distances from the foci is greater, equal, or less, than the transverse axis.

Cor. 3.—The conjugate axis is bisected in the centre.

For (see figure to Def. 1) $EG = AC = EH$, and hence angle $G = H$, and those at C are right angles (III. Def. 3); therefore (Pl. Ge. I. 26) $GC = CH$.

Cor. 4.—The rectangle under the segments, into which the transverse is divided in one of the foci, is equal to the square of the semi-conjugate axis.

For $CG^2 = GF^2 - CF^2 = CB^2 - CF^2 = EB \cdot BF$.

COR. 5.—The square of the distance of the foci is equal to the sum of the squares of the transverse and conjugate axis.

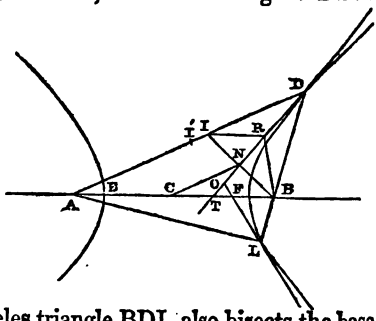
COR. 6.—A perpendicular to the transverse, at its extremity, is a tangent to the hyperbola.

For (Pl. Ge. II. c. Cor. 1) $(AQ + QB)(AQ - QB) = (AF + FB)(AF - FB) = AB \cdot EF$. But $AQ + QB > AB$, therefore $AQ - QB < EF$, and the point Q is without the curve (III. 1, Cor. 2).

PROPOSITION II.

The straight line which bisects the angle, contained by two straight lines, drawn from any point in the hyperbola to the foci, is a tangent to the curve in that point.

Let D be any point in the hyperbola DFL, from which AD, DB, are drawn to the foci, and let the angle BDI be bisected by the line DT, then is DT a tangent to the curve in the point D.



For, let any other point R be assumed in DT, and DI being made equal to DB, let BI, RB, RI, be drawn, and suppose A, R, joined. The line DNT, which bisects the vertical angle of the isosceles triangle BDI, also bisects the base BI at right angles (Pl. Ge. I. 4). Hence $RB = RI$, and $AR - RB = AR - RI$, and therefore less than AI or EF. Consequently, the point R is without the hyperbola.

COR. 1.—The method of drawing a tangent from a given point in the curve, also of drawing a tangent parallel to a line given in position not parallel to the transverse, is evident.

To draw a tangent parallel to a given line; from B draw *BI perpendicular* to the given line, and make $AI = EF$, *then a line*, as NR, bisecting BI perpendicularly, is a tan*gent.*

Cor. 2.—There cannot be more than one tangent to the hyperbola at the same point.

For the difference between the lines AD, DB, that make equal angles with DN, is less than that of any other two lines from A and B to a point in DN. The rest of the proof is similar to that of Cor. 3, Prop. 2, on the ellipse.

Cor. 3.—Every tangent bisects the angle contained by straight lines drawn to the foci from the point of contact; or, which is the same, these lines make equal angles with the tangent.

Cor. 4.—Every tangent to the same hyperbola meets the transverse between its vertex and the centre.

For (Pl. Ge. VI. 3) $AD : DB = AT : TB$; but AD is always greater than DB by EF, therefore AT is always greater than TB.

Cor. 5.—A straight line drawn from the centre to meet a tangent, and parallel to the line joining the point of contact, and one of the foci, is equal to half the transverse axis.

Since $IN = NB$, and $AC = CB$, CN is parallel to AI, and equal to the half of it, or of EF.

Cor. 6.—A perpendicular to a tangent, from one of the foci, and a parallel to the line joining the other focus, and the point of contact from the centre, meet the tangent in the same point.

Cor. 7.—If a chord pass through one of the foci, and the tangents at its extremities be produced to meet, the straight line that joins the point of concourse and the focus, will be perpendicular to the cord.

From O draw a line perpendicular to AD, and if it does not meet it in I, let it meet it in a point I′. Then (Pl. Ge. VI. H) $AL + DI' =$ semi-perimeter of the triangle ADL. But since $AD - DB = EF = AL - LB$, therefore $AL + DB = AD + LB$; and hence $AL + DB =$ semi-perimeter, and therefore $DI' = DB$, and the point I′ must therefore coincide with I. Therefore in the triangles ODI, ODB, the sides OD, DI, are equal to OD, DB, and the contained angles at D are equal; therefore angle $OBD = OID =$ a *right angle;* and OB is perpendicular to DL.

PROPOSITION III.

From any point in an hyperbola, a perpendicular being let fall upon the transverse, and straight lines drawn to the foci; as half the transverse is to the distance of the centre from the focus, so is the distance of the centre from the perpendicular to half the sum of the straight lines drawn to the foci.

Let D be the point in the curve, from which DA, DB, are drawn to the foci, and DK perpendicular to EF; then $CF : CB = CK : \frac{1}{2}(AD + DB)$ (figure Prop. 1).

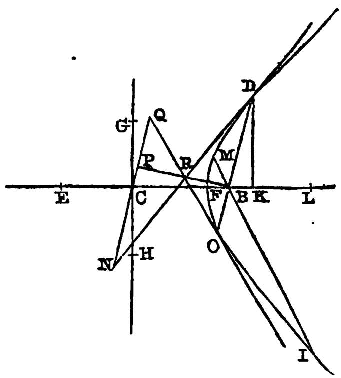
For, let $EFL = AD$, then $LF = DB$, and $2CL = AD + DB$. But $(AD + DB) \cdot (AD - DB) = AB \cdot 2CK$ (Pl. Ge. II. c. Cor. 1), or $EF \cdot 2CL = AB \cdot 2CK$; therefore $EF : AB = 2CK : 2CL$, and $CF : CB = CK : CL$.

PROPOSITION IV.

Any two chords which intersect each other in the focus of an hyperbola, are to one another directly as the rectangles under their segments.

Let the chords DO, MI, intersect one another in the focus B of the hyperbola DFO, $DO : MI = DB \cdot BO : MB \cdot BI$.

For, let the tangents at D, O, meet each other in R, through C let NQ be drawn parallel to DO, meeting the tangents in N, Q, and let BR meet NQ in P. It has been proved that CN or CQ = CF (III. 2, Cor. 5), therefore $NQ = EF$; that $CF : CB = CK : CL$ (III. 3), therefore $CF \cdot CL = CB \cdot CK$, and that DBR is a right angle (III. 2, Cor. 7), therefore the triangles DBK, BCP, are similar, and



$DB \cdot CP = BC \cdot BK$. Hence if $FL = DB$, $CF^2 = CF \cdot CL$

$- CF \cdot FL = BC \cdot CK - DB \cdot CQ = BC \cdot CK - BC \cdot BK - DB \cdot PQ = BC^2 - DB \cdot PQ$, and $DB \cdot PQ = BC^2 - CF^2 = EB \cdot BF$. But $DO : NQ = BO : PQ = DB \cdot BO : DB \cdot PQ$.

Therefore $\quad DO : EF = DB \cdot BO : EB \cdot BF$.

In like manner, $\quad EF : MI = EB \cdot BF : MB \cdot BI$.

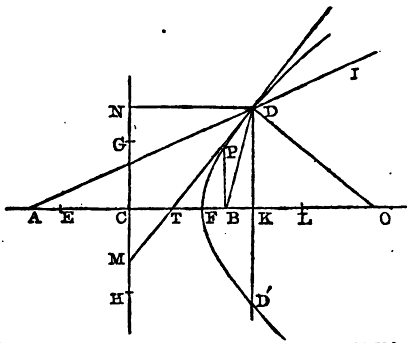
Consequently (Pl. Ge. V, 22) $DO : MI = DB \cdot BO : MB \cdot BI$.

COR.—The rectangle under any chord passing through the focus and the parameter of the transverse, is equal to four times the rectangle under its segments.

For it was proved that $DO : EF = DB \cdot BO : EB \cdot BF = 4DB \cdot BO : 4EB \cdot BF$. Therefore, if P be the parameter of the transverse, $P \cdot DO : P \cdot EF = 4DB \cdot BO : 4EB \cdot BF$. But $P \cdot EF = GH^2 = 4EB \cdot BF$, therefore $P \cdot DO = 4DB \cdot BO$.

PROPOSITION V.

If a tangent meet either axis of the hyperbola, and a perpendicular to that axis be drawn from the point of contact, then shall half the axis be a mean proportional between its segments intercepted from the centre by the tangent and perpendicular.



Case 1. Let the tangent DT, and perpendicular DK, meet the transverse in T; K, $CT : CF = CF : CK$.

For, since DT bisects the angle ADB, $AT : TB = AD : DB$, and mixing and halving the terms,

$\frac{1}{2}(AT + TB) : \frac{1}{2}(AT - TB) = \frac{1}{2}(AD + DB) : \frac{1}{2}(AD - DB)$.

That is, $\quad CB : CT = CL : CF$.

But $\quad CF : CB = CK : CL$ (III. 3).

Therefore $\quad CF : CT = CK : CF$.

And $\quad CT : CF = CF : CK$.

Case 2. Let the tangent DT, and the perpendicular DN, meet the conjugate in M, N; $MC : CG = CG : CN$.

For, let DO, perpendicular to DT, meet the transverse in O, DO evidently bisects the exterior vertical angle of the triangle ADB, therefore $AO : OB = AD : DB = AT : TB$. Hence mixing, halving, and inverting $CB : CO = CT : CB$, and $CT \cdot CO = CB^2$. But $CT \cdot CK = CF^2$, consequently $CT \cdot KO = CB^2 - CF^2 = CG^2$.

Again, from the similar triangles CTM, DKO, $CT \cdot KO = MC \cdot DK = MC \cdot CN$; therefore $MC \cdot CN = CG^2$, and $MC : CG = CG : CN$.

Cor. 1.—The rectangle under the segments of the transverse, into which it is divided by the perpendicular, is equal to the rectangle under the distances of the perpendicular from the centre, and the point where the tangent meets the transverse.

For $EK \cdot KF = CK^2 - CF^2 = CK^2 - CK \cdot CT = CK \cdot KT$.

Cor. 2.—The segments into which either axis is divided by the tangent, are directly proportional to the segments into which it is divided by the perpendicular.

For $CT : CF = CF : CK$, and, by mixing, $ET : TF = EK : KF$.

PROPOSITION VI.

If from any point in an hyperbola, a perpendicular fall upon the transverse axis, as the square of that axis is to the square of the other, so is the rectangle under the segments of the former, to the square of the perpendicular.

Let the perpendicular DK fall from any point D in the curve, upon EF, $FF^2 : GH^2 = EK \cdot KF : DK^2$ (figure to proposition 5).

For, since $CT : CF = CF : CK$ (III. 5), $CF^2 : CK^2 = CT : CK$, and, by conversion, $CF^2 : EK \cdot KF = CT : TK$. But $CT : TK = MT : TD = CM : CN = CG^2 : CN^2$ (III. 5), or DK^2, therefore $CF^2 : EK \cdot KF = CG^2 : DK^2$, and alternately $CF^2 : CG^2 = EK \cdot KF : DK^2$. Consequently $EF^2 : GH^2 : EK \cdot KF : DK^2$.

Cor. 1.—The square of the conjugate axis is to the square of the transverse, as the sum of the squares of

the semi-conjugate, and the distance of the centre from a perpendicular falling upon the conjugate, from any point in the curve, to the square of that perpendicular.

For it has been shown that $DK^2 : CG^2 = EK \cdot KF : CF^2$, and, by addition, $DK^2 : CG^2 + CN^2 = EK \cdot KF : EK \cdot KF + CF^2$, or CK^2. By alternation, $DK^2 : EK \cdot KF = CG^2 + CN^2 : CK^2$. But $CK = DN$, and $DK^2 : EK \cdot KF = CG^2 : CF^2$. Therefore $CG^2 : CF^2 = CG^2 + CN^2 : DN^2$.

Schol.—In the sixth proposition on the ellipse, it appears in the case of a perpendicular on the conjugate axis, that the proportion is exactly similar to the above, with $GN \cdot NH$ or $CN^2 - CG^2$, instead of $CG^2 + CN^2$.

Cor. 2.—The squares of perpendiculars, falling from points in the curve upon its transverse axis, are to one another as the rectangles under the segments of the axis; and the squares of perpendiculars upon the conjugate, are to one another as the sums of the squares of the semi-conjugate, and the distance of each from the centre.

This is proved like the first corollary to the sixth proposition on the ellipse.

Cor. 3.—Every straight line, terminating in the hyperbola, or in opposite hyperbolas, and parallel to one axis, is an ordinate to the other; and conversely.

For, produce DK to D′, and it is similarly shown that $CF^2 : CG^2 = EK \cdot KF : D'K^2$. Therefore $D'K = DK$, and DD′ is an ordinate to EF. By producing DN to cut the opposite hyperbola, it may be similarly shown that the whole line is an ordinate to GH.

Cor. 4.—The two axes are conjugate diameters.

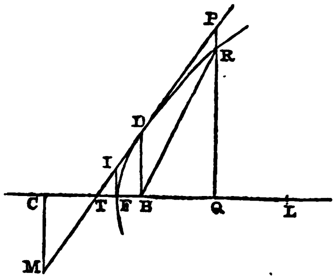
Cor. 5.—The ordinate through the focus is the parameter of the transverse.

For $CF^2 : CG^2 = EB \cdot BF : BP^2$, or $CF^2 : CG^2 = CG^2 : BP^2$, or $CF : CG = CG : BP$.

Cor. 6.—Equal ordinates to either axis are equally distant from the centre, the less ordinate is nearer the centre; and conversely.

PROPOSITION VII.

If through any point in the hyperbola, a parallel to the conjugate axis meet the transverse and the focal tangent, its segment between these will be equal to the distance of that point from the focus.



Through any point R in the curve, let RQ, parallel to MC, meet the focal tangent DM, and transverse axis EF in P, Q; then PQ = RB.

For, let FL = RB. Then, because CT : CF = CF : CB (III. 5) = CQ : CL (III. 3).

Alternately CT : CQ = CF : CL.
By conversion CT : TQ = CF : FL.
Hence MC : PQ = CF : RB.

But (III. 2, Cor. 5) MC = CF, consequently PQ = RB.

Cor. 1.—A perpendicular to the transverse, at one extremity, limited by the focal tangent, is equal to the distance of that extremity from the focus.

For CT : CF = CF : CB; therefore, by conversion, CT : TF = CF : FB. But CT : TF = CM : FI, and CM = CF, therefore FI = FB.

Cor. 2.—The distance of any point in the curve, from the perpendicular to the transverse, at the point where that axis meets the focal tangent, is to its distance from the focus, as half the transverse is to the distance of the centre from the focus.

For CF : CB = CT : CF (III. 5) = CT : CM = TQ : PQ or RB, or TQ : RB = CF : CB.

Schol.—The observations made in reference to the ellipse, in the scholium to the eleventh proposition on that curve, apply to the hyperbola, in regard to the property stated in the above corollary. When the determining ratio of CB to CF is one of minority, the curve is an ellipse; when it

is one of majority, it is a hyperbola; and when of equality, it is a parabola.

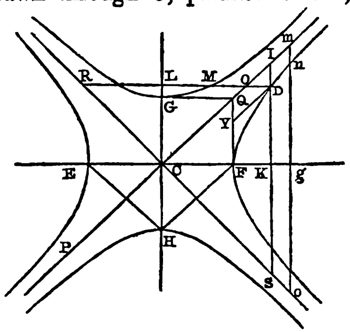
COR. 3.—Every perpendicular to the transverse beyond its vertices, meets the curve on each side of the axis.

For it must meet the focal tangent, as it is inclined to it, and hence it must meet the curve.

PROPOSITION VIII.

Straight lines drawn through the centre, parallel to the lines which join the extremities of the axes, are assymptotes to the hyperbolas.

The lines PQ, RS, drawn through C, parallel to FH, HE, do not meet, but continually approach the curve, so as to come within less than any given distance from it.



For, let FQ be perpendicular to EF, and through the points K, g, assumed in the transverse produced, let IS, m o, be drawn parallel to GH, meeting PQ in I, m, the curve in D, n, and RS, in S, o. Then, since $CF^2 : CG^2 = EK \cdot KF : DK^2$ (III. 6), and FQ = CH, or CG; $CF^2 : FQ^2 = EK \cdot KF : DK^2$. Hence $CK^2 : KI^2 = EK \cdot KF : DK^2$; but CK^2 is always greater than $EK \cdot KF$, therefore KI is greater than DK. Thus, every point of PQ is without the curve.

Again, by alternation and conversion,

$$CK^2 : CF^2 = KI^2 : ID \cdot DS.$$

$$\text{But } CK^2 : CF^2 = KI^2 : FQ^2.$$

Therefore $ID \cdot DS = FQ^2$. In like manner, $mn \cdot no = FQ^2$. Hence $ID : mn = no : DS$, and, since no is greater than DS, ID is greater than mn, whence the point n is nearer to PQ than the point D.

Lastly, to show that the distance of the curve from PQ, *when both are* produced, becomes less than any given line,

however small, let DY be drawn parallel to PQ, at the distance of the given line from it, on the same side with the hyperbola, and meeting FQ in Y; then, to QY and FQ, let DS, parallel to FQ, and terminating in DY, CS, be a third proportional, and let it be produced to meet PQ in I; ID is equal to QY, therefore ID : FQ = FQ : DS, and $ID \cdot DS = FQ^2$. Whence, it is plain, from what has been demonstrated, that D is a point in the curve. Now, D is at the distance of the given line from PQ; consequently the curve, at any point beyond it, is at a less distance from PQ.

COR. 1.—There cannot be more than two assymptotes to the hyperbola; and the same lines are also assymptotes to the other three hyperbolas.

It may be similarly proved that PQ, RS, are assymptotes to the other three hyperbolas.

COR. 2.—The four hyperbolas approach indefinitely near, but do not meet one another.

COR. 3.—Every straight line, passing through the centre, except the assymptotes, meets two opposite hyperbolas, each in one point only.

For if a straight line pass through two points in the curve, as n and D, it will evidently pass below YD, and cannot pass through C.

COR. 4.—If a straight line be drawn through any point in the hyperbola, parallel to either axis, and meeting the assymptotes, the rectangle under its segments from that point is equal to the square of half that axis.

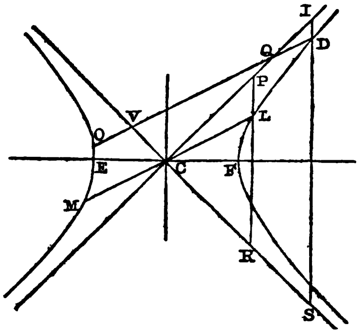
For (III. 6, Cor. 1) $CG^2 : CF^2 = CG^2 + CL^2 : LD^2$, and $GQ = CF$, for $FQ = CG$; and, as it was proved in the demonstration of the proposition that $ID \cdot DS = FQ^2$, it may be shown that $RD \cdot DO = GQ^2$.

COR. 5.—According as the conjugate axis is greater, equal, or less, than the transverse, the angle of the assymptotes is greater, equal, or less, than a right angle.

PROPOSITION IX.

If, from any point in an hyperbola, a straight line be drawn parallel to any diameter, to meet the assymptotes, the

rectangle under its segments, from that point, is equal to the square of half the diameter.



Let D be any point in the hyperbola, and DO drawn parallel to the diameter LCM, meet the assymptotes in Q, V. Then $VD \cdot DQ = \frac{1}{4} LM^2$.

For, through D, and either extremity of LM, let IDS, PLR, be drawn parallel to GH, to meet the assymptotes. Then, because of similar triangles,

$$ID : DQ = PL : LC.$$

And $\quad DS : DV = LR : LC.$

Therefore $ID \cdot DS : DQ \cdot DV = PL \cdot LR : LC^2$. But $ID \cdot DS = PL \cdot LR$ (III. 8, Cor. 4); therefore $DQ \cdot DV = LC^2$. In the same manner, $DV \cdot DQ = CM^2$. Hence $LC = CM$, and $DQ \cdot DV =$ the square of half the diameter LM.

Cor. 1.—Every diameter is bisected in the centre.

Cor. 2.—If a straight line be drawn through two points, in the same, or in opposite hyperbolas, its segments from these points, intercepted by the assymptotes, will be equal.

For, if DV meet the curve in O, $OQ \cdot OV = LC^2 = DV \cdot DQ$, therefore $OQ : DQ = DV : VO$, and, $OD : DQ = OD : VO$. Whence $DQ = VO$, and $DV = QO$.

Cor. 3.—Every tangent limited by the assymptotes is bisected in the point of contact, and is equal to that diameter to which it is parallel. Also, conversely, a straight line, terminated by the assymptotes, and having its middle point in the curve, is a tangent to the hyperbola.

For (figure to proposition 11) $LX = LY$ when D and O coincide (Cor. 2); and since XY is parallel to IN, therefore $CI^2 = LX \cdot LY = LX^2$, and hence $CI = LX$ or $IN = XY$.

Cor. 4.—If, from two points in the same, or in opposite

hyperbolas, parallel straight lines be drawn, to meet the assymptotes, the rectangles under their segments will be equal.

For each of these rectangles is equal to the square of the semi-diameter parallel to it.

PROPOSITION X.

A straight line, terminated by the hyperbola, and parallel to a tangent, is an ordinate to the diameter that passes through the point of contact.

The chord DO, parallel to the tangent XLY, is an ordinate to the diameter LM (figure to proposition 11).

For, let the assymptotes meet the tangent in X, Y, and the chord in Q, V; also, let LM meet DO in R. Then, because XL = LY (III. 9, Cor. 3), RQ = RV. But DQ = OV (III. 9, Cor. 2). Consequently RD = RO.

Cor. 1.—A straight line, terminated by opposite hyperbolas, and parallel to a tangent, is an external ordinate to the diameter that passes through the point of contact.

For PT = SW (III. 9, Cor. 2); and PU = US, for LX = LY; therefore UT = UW.

Cor. 2.—Every ordinate, and every external ordinate, to a diameter, is parallel to the tangent at its vertex.

Cor. 3.—All ordinates to the same diameters are parallel to one another.

Cor. 4.—A straight line, drawn through the vertex of a diameter, parallel to its ordinates, is a tangent to the hyperbola.

Cor. 5.—The diameter which bisects one of two parallel chords, will also bisect the other.

Cor. 6.—The straight line, which bisects two parallel chords, passes through the centre of the hyperbola.

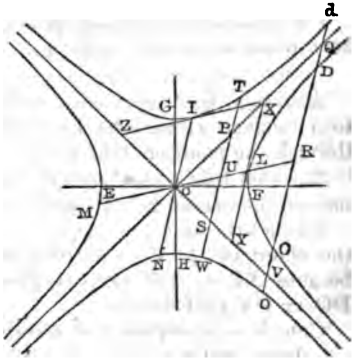
Cor. 7.—If two tangents be at the vertices of the same diameters, they are parallel; and conversely.

PROPOSITION XI.

Two diameters, one of which is parallel to the tangent at the vertex of the other, are conjugate to one another.

Let LCM and ICN be two diameters, of which IN is *parallel to* the tangent XY, at the vertex of LM; IN, LM, *are conjugate* diameters.

For, let XI be drawn and produced to meet the other assymptote in Z. Then, because XL is equal and parallel to IC, XZ is parallel to LM. Now, the triangles ICZ, LCY, by reason of parallel lines, are equiangular, and they have the side IC = LY; therefore IZ = LC = IX. Whence XZ is a tangent at I; and LM has beeen proved parallel to it. Consequently, IN, LM, are mutually parallel to each other's ordinates.



COR. 1.—Tangents to the four hyperbolas, at the vertices of two conjugate diameters, form a parallelogram, whose diagonals are coincident with the assymptotes.

COR. 2.—Any two conjugate diameters of an equilateral hyperbola are equal to one another.

For XCY is in this case a right-angled triangle, and therefore L is the centre of its circumscribing circle, and CL = LX = CI, or ML = IN.

PROPOSITION XII.

If an ordinate be applied to any diameter, the square of that diameter is to the square of its conjugate as the rectangle under its segments to the square of the semi-ordinate.

Let LCM be any diameter, IN its conjugate, and DRO an ordinate to LM, in the hyperbola DLO; $LM^2 : IN^2 = MR \cdot RL : DR^2$ (figure to proposition 11).

For, let DO meet the assymptotes in Q, V, and the tangent at L meet them in X, Y. Then, from similar triangles, $CL^2 : CR^2 = LX^2 : RQ^2$, by conversion and alternation, $CL^2 : LX^2 = MR \cdot RL : RQ^2 - LX^2$. But $LX^2 = CI^2$ (III. 9, Cor. 3) $= QD \cdot DV$ (III. 9), $= RQ^2 - DR^2$, hence $RQ^2 - LX^2 = DR^2$; therefore $CL^2 : CI^2 = MR \cdot RL : DR^2$, and $LM^2 : IN^2 = MR \cdot RL : DR^2$.

COR. 1.—If an external ordinate be applied to any diameter, the square of that diameter is to the square of its conjugate, as the sum of the squares of the semi-diameter and segment intercepted by the external ordinate, from the centre, to the square of the external semi-ordinate.

For, let TW be an external ordinate; then $CL^2 : CU^2 = LX^2 : UP^2$; by addition $CL^2 : CL^2 + CU^2 = LX^2 : LX^2 + UP^2$; by alternation $CL^2 : LX^2 = CL^2 + CU^2 : LX^2 + UP^2$. Now $LX^2 = CI^2 = PT \cdot TS = UT^2 - UP^2$; therefore $UT^2 = LX^2 + UP^2$, and hence $CL^2 : CI^2 = CL^2 + CU^2 : UT^2$.

Schol.—The proof in this corollary is the same as that of the proposition, with the exception that addition is taken instead of conversion.

COR. 2.—The squares of semi-ordinates are to one another as the rectangles under the segments of the diameter.

COR. 3.—Any diameter of an hyperbola is to its parameter, as the rectangle under its segments to the square of the semi-ordinate that divides them.

For, $LM : IN = IN : P$, the parameter; then $LM : P = LM^2 : IN^2 = CL^2 : CI^2 = MR \cdot RL : DR^2$.

COR. 4.—In an equilateral hyperbola, the rectangle under the segments of any diameter is equal to the square of the semi-ordinate that divides them; and the square of an external semi-ordinate is equal to the sum of the squares of the semi-diameter and segment intercepted from the centre.

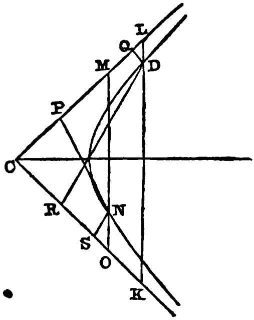
For then $CL = CI$, and therefore $MR \cdot RL = DR^2$, and $CL^2 + CU^2 = UT^2$.

PROPOSITION XIII.

If from any point in an hyperbola, two straight lines be drawn, to meet the assymptotes, and from any other point in the same, or in the opposite hyperbola, straight lines, parallel to these, be also drawn to the assymptotes; the rectangle under the former shall be equal to the rectangle *under the latter.*

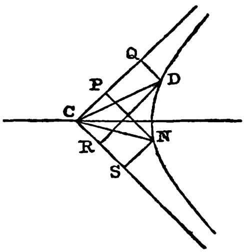
From the point D, in the hyperbola, let there be drawn

ıy two straight lines DQ, DR, to terminate in the assymp-
tes CQ, CS, and from any other
›int N let NP be drawn parallel
DQ, and NS parallel to DR, to
rminate in the same lines. Then
D · DQ = PN · NS.



For, let LDK and MNO be
'awn parallel to the conjugate axis,
meet the assymptotes. Then,
om the similar triangles RDK,
NO, RD : DK = NS : NO, and
om the similar triangles, LDQ,
NP, DQ : LD = PN : MN.
herefore RD · DQ : LD · DK =
N · NS : MN · NO. But LD · DK = MN · NO (III. 9,
or. 4). Consequently RD · DQ = PN . NS.

Cor. 1.—If from two points in the same or in opposite hyperbolas, straight lines be drawn, each parallel to one assymptote, and meeting the other, they are to one another inversely as the parts they intercept from the centre.



For QD, DR, being parallel re-
›ectively to PN and NS, therefore
D · DR = NS · NP, and conse-
uently QD : NS = NP : DR, or
D : NS = CS : CQ.

Cor. 2.—If from any point in a given hyperbola, two straight lines be drawn parallel to the assymptotes, the parallelogram formed thereby is of a given magnitude.

For (Pl. Ge. VI. 23, Cor. 1) QR : PS = (DQ : NS,
)R : NP) = DQ · DR : NS · NP; but DQ · DR = NS · NP,
herefore QR = PS, and any other rectangle similarly formed
s shown in the same manner to be equal to PS.

Cor. 3.—Every sector of an hyperbola is equal to the quadrilateral figure contained by the curve, by one assymptote, and by parallels to the other, through the extremities of the base of the sector.

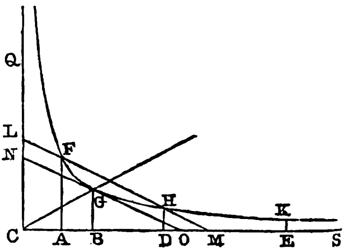
For, since the parallelograms RQ, PS, are equal, the two

triangles CQD, CSN, are together equal to PS, and these equals being taken from the figure CQDNS, there remains the sector CDN equal to the quadrilateral figure PQDN.

PROPOSITION XIV.

If in one of the assymptotes of an hyperbola, any number of points be assumed, such, that their distances from the centre be in continued proportion, and straight lines be drawn from these points to the curve, parallel to the other assymptote, the mixtilineal quadrilateral figures formed thereby will be equal.

Let the points A, B, D, E, be assumed in the assymptote CS, so that CA : CB = CB : CD = CD : CE, and let AF, BG, DH, EK, be drawn to meet the curve parallel to the other assymptote CQ; the quadrilateral figures AFGB, BGHD, DHKE, shall be equal.



For, let the tangent at G, and the line that joins H, F, meet CQ in N, L, and CS in O, M. Then, because OG = GN (III. 9, Cor. 3), OB = BC, and because MH = FL (III. 9, Cor. 2), MD = AC; therefore MD : OB = CB : CD = DH : BG (III. 13, Cor. 1). Hence, the triangles MDH, OBG, which have the angles at D and B equal, are equiangular, and LM is parallel to NO. The diameter CG, therefore, bisects the chord HF, and every chord parallel to HF (III. 10). Consequently CG bisects the segment FGH of the hyperbola. But it also bisects the triangle FCH; therefore the sector CFG = CGH, and the quadrilateral AFGB = BGHD (III. 13, Cor. 3).

Schol.—The ninth and twelfth propositions of the ellipse may be applied to the hyperbola, and demonstrated in the same manner.

EXERCISES.

1. The square of any semi-diameter of an hyperbola is *equal to the* rectangle under the distances of its vertex from

the foci, added to the difference of the squares of the semi-transverse and semi-conjugate axis.

2. Every tangent of an hyperbola is harmonically divided by the transverse axis and perpendiculars falling upon it from the foci.

3. The difference of the squares of any two conjugate diameters of an hyperbola, is equal to the difference of the squares of the two axes.

4. If from any point in an hyperbola, straight lines be drawn through the vertices of a diameter, to limit the tangents at these points, the rectangle under the tangents will be equal to the square of the semi-conjugate diameter.

5. A semi-ordinate to any diameter is a mean proportional between its segments, intercepted from the diameter by two straight lines intersecting each other in any point of the curve, and passing through the vertices of the diameter.

6. If a quadrilateral figure be formed by tangents to the four hyperbolas, a straight line through the centre, parallel to that which joins two opposite points of contact, will divide the two opposite sides of the figure, so that the segments of the one shall be inversely proportional to the segments of the other.

7. Also, the straight line which joins the middle points of its diagonals will pass through the centre of the hyperbolas.

8. If through a fixed point, any straight line be drawn, to meet the hyperbola, or opposite hyperbolas, in two points, the rectangle under its segments, from the fixed point, will be to the rectangle under its segments, intercepted by the assymptotes, from either point in the curve, in a constant ratio.

9. If from a point in one of the assymptotes of an hyperbola, any straight line be drawn to intersect the curve (or opposite curves), in two points, and from the points of section, lines parallel to the same assymptote be drawn to meet the other, the sum (or difference) of the parallels will always be of the same magnitude.

The propositions in the exercises to the preceding book may be applied to the hyperbola.

FOURTH BOOK.

DEFINITIONS.

1. Let there be a circle, and a fixed point, without its plane, and let a straight line, always passing through that point, and indefinitely extended both ways, revolve about the circle in its circumference; the two surfaces thus described are each of them named a *conic surface*, the fixed point its *vertex*, the circle its *base*, and the straight line passing through the vertex and the centre its *axis* (first figure to proposition 1).

2. A *cone* is a solid bounded by a circle and a conic surface. The fixed point is called the *vertex*, and the circle the *base* of the cone. Also, the straight line passing through the fixed point, and the centre of the circle, is named the *axis* of the cone.

3. Let there be a circle, and any straight line intersecting its plane at the centre, and let another straight line, always parallel to the former, revolve about the circle in its circumference; the surface thus described is named a *cylindric surface*, of which the circle is the *base*, and the straight line through its centre the *axis* (second figure to proposition 1).

4. A *cylinder* is a solid bounded by two equal and parallel circles, and a cylindric surface. Either of the circles is named the *base*, and the straight line joining their centres the *axis* of the cylinder.

5. A cone or cylinder is termed *right* or *oblique*, according as the axis is perpendicular or oblique to the base.

6. The section of a cone or cylinder is termed *parallel* or *oblique*, according as its plane is parallel or inclined to the base.

7. When a cone or cylinder is cut by a plane touching the axis, and perpendicular to the base, and by another plane perpendicular to the former, in such a manner, that the common section of the two planes make angles with the common sections of the first plane and the surface, alter-

nately equal to those which the common section of the first plane and the base makes with the same lines on the same side, the section of the second plane is called a *subcontrary section* (figures to Prop. 2).

COROLLARIES FROM THE DEFINITIONS.

Cor. 1.—A straight line joining the vertex, and any point in a conic surface, lies wholly in that surface; its continuation one way is in the same, and its continuation the other way in the opposite surface; also, every such line meets the circumference of the base.

Cor. 2.—Any plane touching the axis of a conic surface, cuts that surface in two straight lines, as AD, AF (fig. to Prop. 1).

Cor. 3.—If a cone be cut by a plane touching the axis, or by a plane through the vertex, and any two points in the circumference of the base, the section is a triangle, as AFD (first figure to Prop. 1).

Cor. 4.—If a plane, touching a tangent to the base, pass through the vertex, it touches the conic surface in the straight line joining the vertex and the point of contact, every point in the plane, except in that straight line, being without the surface.

Cor. 5.—A straight line, drawn from any point in a cylindric surface, parallel to the axis, lies wholly in that surface.

Cor. 6.—Any plane touching the axis of a cylindric surface, cuts that surface in two parallel straight lines.

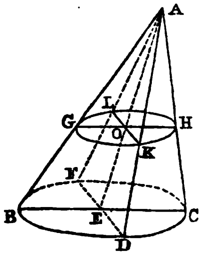
PROPOSITION I.

If a conic or cylindric surface be cut by a plane parallel to the base, their line of common section is the circumference of a circle, having its centre in the axis.

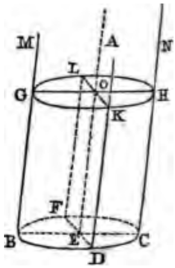
1. Let ABDCA be a conic surface, of which BCD is the base, and AE the axis, and let it be cut by the plane GOL, parallel to BCD, the line of common section GKHL is the circumference of a circle having its centre in AE.

For, let any two planes, ABC, AFD, touching the axis AE, *cut the* surfaces in the straight lines AB, AC, AF,

AD (Cor. 2 to Def.); the base BCD, in the diameters BEC, DEF; and the parallel section GOL, in the straight lines GOH, KOL. Then BC is parallel to GH, and FD to KL. Hence, by similar triangles, ED : OK (= AE : AO) = EC : OH. But ED = EC, therefore OK = OH. Consequently all straight lines drawn from the point O, where the axis meets the parallel plane GOL, to terminate in the line of common section GKHL, are equal to one another, and GKHL is the circumference of a circle, of which O is the centre.



2. Let MBDCN be a cylindric surface, of which BCD is the base, and AE the axis, and let it be cut by the plane GOL, parallel to BCD, the line of common section GKHL is the circumference of a circle, having its centre in AE.



For, let any two planes, ABC, AFD, touching the axis AE, cut the surface in the straight lines MB, NC; LF, KD (Cor. 6 to Def.), the base BCD, in the diameters BEC, DEF, and the parallel section GOL, in the straight lines GOH, KOL. Then BC is parallel to GH, and FD to KL. But NC, KD, are each parallel to AE. Therefore OD, OC, are parallelograms, and OK = ED = EC = OH. Whence GKHL is the circumference of a circle, of which O is the centre.

Cor. 1.—If a conic surface be cut by a plane, parallel to the base, the solid betwixt that plane and the vertex is a cone.

Cor. 2.—If a cylindric surface be cut by two planes parallel to the base, the solid betwixt them, and the solids between each of them and the base, are cylinders.

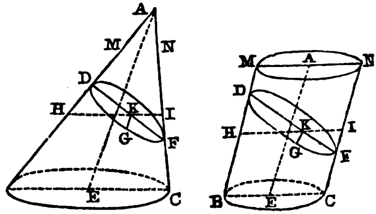
Cor. 3.—If a cone or cylinder be cut by a plane parallel to the base, the section is a circle.

Cor. 4.—Any plane touching the axis of a cone or cylinder, cuts every parallel section in its diameter.

PROPOSITION II.

Every subcontrary section of an oblique cone or cylinder is a circle.

Let the cone or cylinder MBCN be cut by the plane MC, perpendicular to the base, touching the axis, and meeting the surface in the straight lines MB, NC, and the base in the diameter BC. Let it be cut by another plane DFG, perpendicular to the former, so that their line of common section DF may form, with one of the lines MB, NC, the angle NFD, equal to the angle MBC, which BC forms with the other on the same side. The latter section DGF, which is called a subcontrary section, is a circle, and DF its diameter.



For, let the parallel section HGI pass through any point K in DF. Because HGI is parallel to the base, it is perpendicular to MC, and therefore GK (So. Ge. I. 18) its line of common section, with DGF, is perpendicular to the same. Thus, GK is at right angles to DF and HI, and since HI is a diameter of the parallel section (IV. 1, Cor. 4), the rectangle $HK \cdot KI = GK^2$. But the triangles HDK, IKF, being similar (for the angles at K are equal, and the angle $NFD = MBC = DHK$), $DK : HK = KI : KF$, and the rectangle $DK \cdot KF = HK \cdot KI$. Consequently the rectangle $DK \cdot KF = GK^2$, and the section DGF a circle, having DF for a diameter.

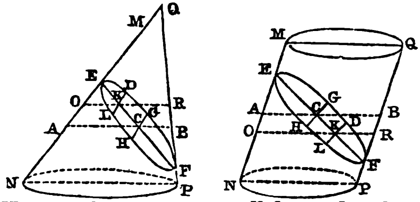
COR.—Every subcontrary section of an oblique cylinder is equal to its base.

For the triangles HDK, IKF, are isosceles.

PROPOSITION III.

Every section of a cone or cylinder, by a plane meeting the conic or cylindric surface on every side, that is neither *a parallel nor* a subcontrary section, is an ellipse.

Let EHFG be a section of a cone or cylinder MNP, by a plane meeting the surface on every side, but neither a parallel nor a subcontrary section, EHFG is an ellipse.



For, let C be the middle of EF, and K any other point in it; through C and K let planes pass parallel to the base cutting MNP in AB, OR, and EHFG in HG and LD. Also, let the plane MNP cut the base in a diameter NP perpendicular to the line of common section of the plane of the base and of the section EHG. Then HG is parallel to that line of common section (So. Ge. I. 14), and AB to NP; therefore AB is also perpendicular to HG (So. Ge. I. 9). For a similar reason, OR is perpendicular to LD. Now (IV. 1), the sections AHBG, OLRD, are circles, of which AB, OR, are diameters, and therefore $AC \cdot CB = CG^2$, and $OK \cdot KR = DK^2$.

By similar triangles $EC : EK = AC : OK$.
And $CF : KF = CB : KR$.
Therefore $EC^2 : EK \cdot KF = CG^2 : DK^2$.

Consequently EHFG is an ellipse, of which EF and GH are two conjugate diameters.

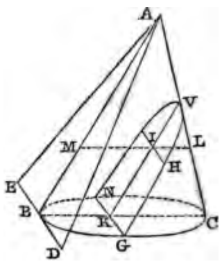
PROPOSITION IV.

If a cone be cut by a plane parallel to another, touching the conic surface, the section is a parabola.

Through any point B, in the circumference of the base BGC, let the tangent DE be drawn, the plane ADE, touching that line, and the vertex A, touches the conic surface in the straight line AB (Cor. 4 to Def.). Let the cone be cut by the plane VGN, parallel to ADE, and meeting the base in GN; the section GVN is a parabola.

For, let ABC, the plane of AB and the axis, intersect *GVN, in the* line VK, and the base in the diameter

BKC. Also, let MHL be a parallel section, through any point I in VK, intersecting ABC, in the diameter ML, and VKG in the straight line HI. Then, by reason of parallel planes, the straight line AB is parallel to VK, BD to GK, GK to HI, and BC to ML (So. Ge. I. 14). Hence MK is a parallelogram; and since DBC is a right angle, GK is perpendicular to BC, and HI to ML

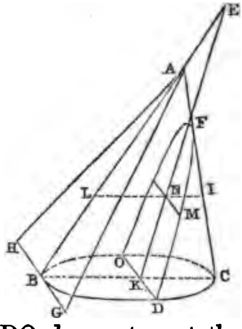


(So. Ge. I. 9). Now, by similar triangles, VK : VI = KC : IL; but BK = MI, therefore VK : VI = BK · KC : MI · IL = $GK^2 : HI^2$. Whence GVN is a parabola, of which VK is a diameter, and GK its semi-ordinate.

PROPOSITION V.

Every other section of a cone is a hyperbola.

Let DFO be a section of the cone ABDC, different from any that has been mentioned, meeting the base in the line DO. Let DO be cut at right angles by the diameter BC, and let ABC, the plane of the axis, and BC, intersect the surface, and the plane FDO, in the straight lines AB, AFC, and FK, and let AHG be a tangent plane through AB. Then FK is not parallel to AB; otherwise, as DK is parallel to the tangent BG, the plane FDO would be parallel to the tangent plane AGB (So. Ge. I. 13), contrary to the hypothesis; let them meet, therefore, in the point E, which will be in the opposite surface, because, by hypothesis, the plane FDO does not meet the



other on every side. Also, let LMI be a parallel section, through any point N in FK, intersecting ABC in the diameter (IV. 1, Cor. 4) LI, and DKF, in the straight line *MN*. Then LI is parallel to BC, and MN to DK (So. Ge.

I. 14), and therefore MN perpendicular to LI (So. Ge. I. 9). Now, by similar triangles,

$$EK : EN = BK : LN.$$

And $$KF : NF = KC : NI.$$

Therefore $EK \cdot KF : EN \cdot NF = BK \cdot KC : LN \cdot NI = DK^2 : MN^2$. Whence DFO is a hyperbola, of which EF is a diameter, and DK, MN, semi-ordinates

THE END.

Edinburgh: Printed by W. & R. Chambers,
19, Waterloo Place.